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ELEMENTS

OF

ALGEBRA.

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IN

BOWDOIN COLLEGE.

1st EDITION

PORTLAND—PUBLISHED BY SHIRLEY & HYDE.

GRIFFIN, PRINTER—BRUNSWICK.

1830.

NOTICE.

The design of the writer of the following treatise has been chiefly to provide a convenient text book for his own classes. In the preparation of the work free use has been made of the Algebra of Lacroix and of Bourdon. The whole has however been written anew, and the subjects are in general presented in such a manner, as has been found by experiment to be well adapted to the business of elementary instruction. It is proposed to continue the subject in another volume, which will contain the theory of the higher equations, and also the application of Algebra to Geometry.

Bowd. Coll. Jan. 1830.

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ELEMENTS OF ALGEBRA.

SECTION I. EXPLANATION OF ALGEBRAIC SIGNS.

Art. 1. In the solution of a question in numbers two distinct things are to be attended to; 1°. to determine the operations to be performed; 2°. to perform these operations.

Let it be proposed, for example, *to divide the number* 18 *into two such parts, that the greater may exceed the less by* 4.

To resolve this question, we remark that,

1°. *The greater part is equal to the less added to* 4,

2°. *The greater part, added to the lesser part, is equal to* 18.

It follows therefore that,

3°. *The lesser part, added to* 4, *added also to the lesser part, is equal to* 18.

But this language may be abridged, thus,

4°. *Twice the lesser part, added to* 4, *is equal to* 18; whence

5°. *Twice the lesser part is equal to* 18 *diminished by* 4.

Subtracting therefore 4 from 18, we have

6° *Twice the lesser part equal to* 14; wherefore

7°. *Once the lesser part is equal to* 14 *divided by* 2, or performing the division, we have

8°. *Once the lesser part equal to* 7.

Adding 4 to 7 we have 11 for the greater part. The parts required therefore are 7 and 11.

Art. 2. In the process of reasoning required in the solution of the proposed question expressions, such as "added to," "di-

minished by," "equal to" &c. are often repeated. These expressions refer to the operations, by which the numbers given in the question are connected among themselves, or to the relations, which they bear to each other. The reasoning therefore, which pertains to the solution of a question, it is evident, may be rendered much more concise, by representing each of these expressions by a convenient sign.

Art 3. It is agreed among mathematicians to represent the expression "added to" by the sign $+$ read *plus*, the expression "diminished by" by the sign $-$ read *minus*, the expression "multiplied by" by the sign $\times$, that of "divided by" by the sign $\div$. Lastly the expression "equal to" is represented by the sign $=$.

Art 4. By means of the above signs the reasoning in the question proposed may be much abridged; still however we have frequent occasion to repeat the expression "the lesser part." The reasoning therefore may be still more abridged by representing this also by a sign.

The lesser part is the unknown quantity sought directly by the reasoning pursued. It is agreed in general to represent the unknown quantity or quantity sought in a question by some one of the last letters of the alphabet, as x, y, z.

Art 5. Let us now resume the question proposed and employ in its solution the signs, which have been explained.

Let us represent by x the lesser of the two parts required, we have then

$$x + 4 = \text{the greater part,}$$
$$x + 4 + x = 18$$
$$2 \times x + 4 = 18$$
$$2 \times x = 18 - 4$$
$$2 \times x = 14$$
$$x = 14 \div 2$$
$$x = 7$$

The multiplication of x by 2 may be expressed more concisely thus, $2 . x$, or still more simply thus, $2 x$. Division also is more commonly indicated by writing the number to be divided above a horizontal line, and the divisor beneath it

in the form of a fraction; 14 divided by 2, for example, is indicated, thus, $\frac{14}{2}$.

Art 6. The question, which we have solved, is simple; it is sufficient, however, to show the aid, which may be derived from convenient signs in facilitating the reasonings, which pertain to the solution of a question. Indeed in abstruse and complicated questions, it would often be difficult, and sometimes absolutely impossible to conduct, without such aid, the reasonings required.

Art 7. The signs which have been explained, together with those which will hereafter be introduced, are called *Algebraic signs*. It is from the use of these that the science of *Algebra* is derived.

Let us now employ the signs already explained in the solution of some questions.

1°. Three men A, B and C trade in company and gain \$406, of which B has twice as much as A, and C three times as much as B. Required the share of each.

Let x represent the share of A, then $2\,x$ will represent the share of B and $6\,x$ the share of C. Then since the shares added together should be equal to the sum gained, we have

405

$$x + 2\,x + 6\,x = 405$$
$$9\,x = 405$$
$$x = \frac{405}{9} = 45$$

Thus we have A's share = \$45; whence B's share is \$90 and C's \$270.

2°. A gentleman meeting four poor persons distributed 5 shillings among them; to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

3°. To divide the number 230 into three such parts, that the excess of the mean above the least may be 40, and the excess of the greatest above the mean may be 60.

Let x represent the least part, then $x + 40$ will be the

mean, and $x + 40 + 60$ will be the greatest part; we have therefore

$$\begin{aligned} x + x + 40 + x + 40 + 60 &= 230 \\ 3x + 140 &= 230 \\ 3x &= 90 \\ x &= 30 \end{aligned}$$

The parts will then be 30, 70 and 130 respectively.

4. A draper bought three pieces of cloth, which together measured 159 yards. The second piece was 15 yds. longer than the first, and the third 24 yds. longer than the second. What was the length of each ?

5. Three men A, B and C made a joint stock; A puts in a certain sum, B puts in \$115 more than A, and C puts in \$235 more than B; the whole stock was \$1753. What did each man put in ? A. 429⅓. B = 544⅓. C = 779⅓.

6. A cask which held 146 gallons was filled with a mixture of brandy, wine and water. In it there were 15 gallons of wine more than there were of brandy, and 25 gallons of water more than there were of wine. What quantity was there of each ? x = Brandy = 30⅓. Wine = 45⅓. Water = 70⅓

7. A gentleman buys 4 horses, for the second of which he gives £12 more than for the first, for the third £6 more than for the second, and for the fourth £2 more than for the third. The sum paid for all was £230. How much did each cost ? 1st = 45. 2d = 57. 3d = 63. 4th = 65£.

8. The sum of \$300 was divided among 4 persons; the second received three times as much as the first, the third as much as the first and second, and the fourth as much as the second and third. What did each receive ? 20, 60, 80, 140.

9. To find a number such, that one half and one third of this number will be equal to 30.

Let x represent the number sought, then one half of the number will be represented by $\frac{1}{2}x$ or $\frac{x}{2}$ and one third by $\frac{1}{3}x$ or $\frac{x}{3}$, we have then

$$\frac{x}{2} + \frac{x}{3} = 30$$

Reducing the fractions $\frac{x}{2}$, $\frac{x}{3}$ to a common denominator, we have

$$\frac{3x}{6} + \frac{2x}{6} = 30$$

whence $$\frac{5x}{6} = 30$$

and $$x = 36$$

10°. Required a number, to which if $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of itself be added the sum will be 50. x = 24 the no.

11°. To find a number such, that twice the number will exceed $\frac{4}{5}$ of it by 40. 33.3

Section II. EQUATIONS.

Art. 8. The solution of a question by the aid of algebraic signs, it will be perceived, is composed of two distinct parts. In the first, we express by these signs the relations established by the nature of the question between the known and unknown quantities. We thus obtain an expression for the equality of two things among themselves. This is called an *equation*. In the second, we deduce from the equation of the question a series of other equations, in the last of which the value of the unknown quantity is determined.

The process of forming an equation, by means of the conditions of the question, is called *putting the question into an equation.*

The process, for deducing the value of the unknown quantity from the equation of the question, is called *resolving or reducing the equation.*

The quantity or quantities on one side the sign = in an equation are called a *member*; an equation has two members. The one on the left of the sign of equality is called the *first* member, and the other the *second.*

The quantities, which compose a member, when separated by the signs + and —, are called *terms.*

Thus in the equation $7x + 5x = 25$ the expression $7x + 5x$ is the first member, and 25 the second.

The quantities $7x$ and $5x$ are the terms of the first member.

A figure written before a letter, showing how many times the letter is to be taken, is called the *coefficient* of that letter. In the quantities $7\,x, 5\,x$, 7 and 5 are the coefficients of x.

Equations are distinguished into different degrees. An equation, in which the unknown quantity is neither multiplied by itself, nor by any other unknown quantity, is called an equation of *the first degree.*

9. No general and exact rule can be given for putting a question into an equation. When however the equation of a question is formed, there are regular steps for its reduction, which we shall now explain.

In order to this we remark, *that when equal operations are performed upon equal quantities the results will be equal.* This is self evident. It follows therefore, since the two members of an equation are equal quantities that, 1°. *the same quantity may be added to both sides of an equation without destroying the equality*; 2°. *the same quantity may be subtracted from both sides of an equation without destroying the equality*; 3°. *both sides of an equation may be multiplied, or* 4°. *both sides may be divided by the same quantity without destroying the equality.*

10. Let it be proposed to resolve the equation $25\,x = 125$. Dividing both sides by 25, we have $x = 5$.

11. Let there be proposed the equation $\frac{x}{5} = 20$. Multiplying both sides by 5, we have $x = 100$.

12. Let us take next the equation $3\,x + 25 = 60 - 4\,x$.

To resolve this equation, it will be necessary to transfer the terms 25 and $4\,x$ from the members, in which they now stand, to the opposite. In order to this, let us first subtract 25 from both members, we then have

$$3\,x + 25 - 25 = 60 - 4\,x - 25$$

or

$$3\,x = 60 - 4\,x - 25$$

Adding next $4\,x$ to both sides of this last, we have

$$3\,x + 4\,x = 60 + 4\,x - 4\,x - 25$$

or

$$3\,x + 4\,x = 60 - 25$$

Comparing the last equation with the proposed, the following rule, for transposing a term from one member of an equation to the other, will be readily inferred, viz. *Efface the term in the member in which it stands, and write it in the other with the contrary sign.*

Art 13. Let there now be proposed the equation $\frac{x}{3}+\frac{x}{4}=20$. Multiplying first by 3, we have

$$x+\frac{3x}{4}=60$$

multiplying this last by 4, we have

$$4x+3x=240$$

Whence to free any term in an equation of its divisor; *Multiply both sides of the equation by the divisor of this term.*

Let us take for a second example the equation

$$\frac{x}{2}+\frac{x}{3}-\frac{x}{6}=7$$

Here the denominator 6 is a multiple of the denominators 2 and 3 ; multiplying therefore first by 6, we have

$$\frac{6x}{2}+\frac{6x}{3}-\frac{6x}{6}=42$$

whence dividing the numerator of each term by its denominator, we have $\qquad 3x+2x-x=42.$

Let us take for a third example the equation

$$\frac{x}{2}+\frac{x}{10}+\frac{3x}{4}-\frac{x}{5}+6=9$$

The least number divisible by each one of the denominators of the proposed, it is easy to see, is 20. If both sides be multiplied by 20, the numerators of the terms which are fractional will be rendered divisible by their denominators ; the equation may then be freed at once from its denominators by performing upon each term the division required.

Multiplying both sides of the proposed by 20, we have

$$\frac{20x}{2}+\frac{20x}{10}+\frac{60x}{4}-\frac{20x}{5}+120=180$$

Whence performing upon each term the division indicated, we have

$$10x+2x+15x-4x+120=180$$

To free an equation therefore in the most simple manner from its denominators; ***Find the least common multiple of the denominators, multiply each term by this common multiple, and divide in the case of terms, which are fractional, the numerator of the term by its denominator.***

Art 14. Let it be proposed next to resolve the equation,

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}.$$

Freeing from denominators, we have

$$10x - 32x - 312 = 21 - 52x$$

transposing and reducing, we have

$$30x = 333$$

whence dividing by 30 we obtain

$$x = 11\tfrac{1}{10}$$

15. The unknown quantity in equations of the first degree can be combined with those which are known in four different ways only, viz. by addition, subtraction, multiplication and division. From what has been done, we have therefore the following rule for the resolution of equations of the first degree with one unknown quantity viz. 1°. ***Free the proposed equation from its divisors***; 2°. ***bring all the terms, which contain the unknown quantity into the first member and all the known quantities into the other***; 3°. ***unite in one term the terms which contain the unknown quantity, and the known quantities in another***; 4°. ***divide both sides by the coefficient of the unknown quantity.***

16. Applying the above rule to the equation

$$\frac{x}{6} - \frac{x}{4} + 10 = \frac{x}{3} - \frac{x}{2} + 11, \text{ we obtain } x = 12.$$

In order to verify this result we substitute 12 for x in the proposed, it then becomes

$$\frac{12}{6} - \frac{12}{4} + 10 = \frac{12}{3} - \frac{12}{2} + 11$$

whence performing the operations indicated we obtain

$$9 = 9$$

The value $x = 12$ satisfies therefore the proposed equation.

In general, to verify the value of the unknown quantity deduced from an equation, we substitute this value for the unknown quantity in the equation. If this renders the two members identically the same, the answer is correct.

17. The following examples will serve as an exercise for the learner in the reduction of equations.

1. $\frac{x}{2}+\frac{x}{3}+\frac{x}{5}=31$

2. $x-7=\frac{x}{5}+\frac{x}{3}$

3. $\frac{x}{4}+\frac{x}{6}-\frac{x}{10}=\frac{57}{4}$

4. $3x+4-\frac{x}{3}=46-2x$

5. $\frac{2x}{3}+4=\frac{7x}{12}+9$

6. $\frac{8x}{5}-11=\frac{9x}{10}-4$

7. $\frac{7x}{3}+5x+\frac{2}{3}=28+\frac{5x}{7}-\frac{1}{2}$

8. $\frac{3x}{8}-\frac{21}{3}=39-5x+\frac{x}{3}-\frac{5}{8}$

9. $\frac{2x}{3}+4=\frac{4x}{5}+12-\frac{5x}{7}$

18. The equations above have been taken at random. It is to be observed, however, that an equation may always be considered as derived from the enunciation of some question. Thus the first equation in the preceding article may be considered as derived from the following enunciation, viz. *to find a number such that one half, one third and one fifth of this number may together be equal to* 31.

19. Though no general and exact rule can be given for putting a problem into an equation, yet the following precept will be found very useful for this purpose, viz. *Indicate by the aid of algebraic signs upon the unknown and known quantities the same reasonings and the same operations, that it would be necessary to perform, in order to verify the answer, if it were known.*

Let us illustrate this precept by some examples.

1. A cistern is supplied by two pipes, the first will fill it alone in 3 hours, the second in 4 hours. In what time will the cistern be filled if both run together?

If the time were known, we should verify it by calculating what part of the cistern would be filled by each pipe separately, these parts added together would be equal to the whole cistern. To indicate the same operations by the aid of algebraic signs, let $x =$ the time, and let the capacity of the cistern be represented by 1. It is evident, that if one of the pipes will fill the cistern in three hours, in one hour it will fill $\frac{1}{3}$ of it, in x hours it will fill x times as much, that is, a part denoted by $\frac{x}{3}$. In like manner in the time x, the second pipe will fill a part denoted by $\frac{x}{4}$; since then these two parts should be equal to the whole cistern, we have for the equation of the problem

$$\frac{x}{3} + \frac{x}{4} = 1$$

from which we obtain $x = 1\frac{5}{7}$ hours.

2. A gentleman distributing money wanted 10 shillings to be able to give 5 shillings to each person; he therefore gave each 4 shillings only and found that he had 5 shillings left. Required the number of persons.

In order to verify the answer if it were known, we should multiply it first by 5 and from the product subtract 10; we should next multiply it by 4 and add 5 to the product. The results thus obtained would be equal to each other, if the answer were correct.

Let us indicate the same operations by the aid of algebraic signs. Putting x for the number of persons sought and multiplying x by 5 we have $5x$, subtracting 10 from this we have $5x - 10$; again x multiplied by 4 gives $4x$, adding 5 to this we have $4x + 5$. Then as these two results should be equal, we have for the equation of the problem

$$5x - 10 = 4x + 5$$

which being resolved gives $x = 15$.

3.° To divide the number 247 into three parts, which may be to each other as the numbers 3, 5 and 11.

If one of the parts, the first for example, were known, we should verify it thus. We should find a number, which would be to this part in the ratio of 3 to 5, this would be 5 to 3, the second part; we should find also a number which would be to the same part in the ratio of 3 to 11, this would be 11 to 3, the third part; the sum of these parts would then be equal to 247.

To imitate this process let $x =$ the first part, the second will then be $\frac{5x}{3}$, and the third $\frac{11x}{3}$. We have then for the equation of the question

$$x + \frac{5x}{3} + \frac{11x}{3} = 247$$

whence $$x = 39$$

4.° A laborer was hired for 48 days, for each day that he wrought he was to receive 24 shillings, but for each day that he was idle he was to forfeit 12 shillings. At the end of the time he received 504 shillings. How many days did he work and how many was he idle?

To verify the numbers required in this problem we should multiply them if known by 24 and 12 respectively; subtracting the last product from the first the remainder would be 504. To indicate these operations by the aid of algebraic signs let $x =$ the number of days in which the laborer wrought, then $48 - x$ will be the number of days, in which he was idle; $24x$ will be the sum due for the number of days in which he wrought, and $48 - x$ multiplied by 12 will be the sum which he forfeited.

The multiplication of $48 - x$ by 12 is indicated by inclosing this quantity in a parenthesis and writing the 12 outside, thus, $12(48 - x)$; we have then for the equation of the question

$$24x - 12(48 - x) = 504$$

To perform the operations indicated in this equation, it is necessary first to multiply $48 - x$ by 12, and then to sub-

tract the result from $24x$. With respect to the multiplication required, since 48 is to be diminished by the number of units in x, it is evident, that 48 multiplied by 12 will be too great for the product required by the number of units in x multiplied by 12; to obtain the true product therefore, from 48×12 we must subtract $x \times 12$; we have then

$$12(48 - x) = 576 - 12x.$$

The subtraction of $576 - 12x$ from $24x$, the next operation required, is indicated by inclosing this quantity in a parenthesis and writing the sign — before it, thus,

$$24x - (576 - 12x).$$

To perform this operation, it is evident, since 576 is to be diminished by $12x$, if we take 576 from $24x$ we take away too much by $12x$; $12x$ must therefore be added to this result in order to obtain the true remainder; we have then

$$24x - (576 - 12x) = 24x - 576 + 12x$$

the equation of the problem therefore becomes

$$24x - 576 + 12x = 504$$

from which we deduce $\qquad x = 30$

20. The following examples are designed as an exercise for the learner in putting a problem into an equation.

1. Divide the number 197 into two such parts, that four times the greater may exceed five times the less by 50.

2. A company settling their reckoning at a tavern pay 8*s*. each; but if there had been 4 persons more they should only have paid 7*s*. each. How many were there?

3. Two workmen received the same sum for their labor; but if one had received 15*s*. more, and the other 9*s*. less, then one would have had just three times as much as the other. What did they receive?

4. Divide the number 56 into two such parts, that one part being divided by 7 and the other by 3, the quotients may together be equal to 10.

5. A gentleman having a piece of work to do hired three men to do it; the first could do it alone in 7 days, the second in 9, the third in 15 days. How long would it take the three together to do it?

6. A cistern is furnished with three cocks, the first will fill it in 5 hours, the second in 13 hours and by the third it would be emptied in 9 hours. In what time will the cistern be filled if all three run together?

7. It is required to divide 25 into two such parts, that the greater may contain the less 49 times.

8. There is a number from which if 11 be subtracted, and the remainder be multiplied by 22, then if we add 29 to this product and divide the sum by 19, the quotient will be equal to the number itself. Required the number.

9. To find two numbers such, that if 6 be added to the first, it will be three times as much as the second, and if 13 be added to the second, it will be twice as much as the first.

10. A father is triple the age of the son; but in 20 years the father will be only double his son's age. Required the age of each.

11. To find a number such, that if 7 be subtracted from it, three fourths of the remainder will be 12.

12. A gentleman gave to three persons 98 pounds. The second received five-eighths of the sum given to the first, and the third one-fifth of what the second had. What did each receive?

13. A father being questioned as to the age of his son replied, that if from double his present age, the triple of what it was six years ago were subtracted, the remainder would be exactly his present age. Required his age.

14. A father intends by his will that his three sons should share his property in the following manner. The eldest is to receive 100 pounds less than half the whole property, the second is to receive 80 pounds less than a third of the whole property, and the third is to have 60 pounds less than a fourth of the property. Required the amount of the whole property, and the share of each son.

15. A person paid a bill of £50 with half guineas and crowns, using in all 101 pieces. How many pieces were there of each, the guineas being reckoned at 21s., and the crowns at 5s.?

16. What two numbers are as 2 to 3, to each of which if 4 be added the sums will be as 5 to 7? $x = 24$ ... $= 16$

17. Divide the number 49 into two such parts, that the greater increased by 6 may be to the less diminished by 11 as 9 to 2. $x = 30$ ... $49 - x$ ...

18. A and B engaged in trade, A with £240, and B with £96. A lost twice as much as B, and upon settling their accounts it appeared that A had three times as much remaining as B. How much did each lose? $x = 48$ ...

19. A has three times as much money as B; but if A gains \$50 and B loses \$93, then A will have five times as much money as B. How much had each? $x = 2$ ...

20. A person has two boxes. Now when he puts 8*s.* into the first it is then half the value of the second; but when he takes the 8*s.* out of the first and puts it into the second, then the second is worth three times as much as the first. What is the value of each? $x =$ ... $+16 = 112$

21. A and B begin trade, A with triple the stock of B. They gain each £50, which makes their stocks in the proportion of 7 to 3. Required their original stocks.

22. A, B and C make a joint stock. A puts in £60 less than B, and £68 more than C, and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4. What did each put in? $x = 200$ ... 110 ...

23. A cistern, into which water was let by two cocks A and B, will be filled by them both running together in 12 hours, and by the cock A alone in 20 hours. In what time will it be filled by the cock B alone?

24. A man being at play lost one fourth of his money and then won 3 shillings; after which he lost one third of what he then had and won 2 shillings; lastly he lost one seventh of what he then had; this being done he had but 12 shillings left. What had he at first?

25. There are three pieces of cloth, whose lengths are in the proportion of 3, 5, and 7; and 6 yards being cut off from each, the whole quantity is diminished in the proportion of 20 to 17. Required the length of each piece at first.

Art. 20 26. Three men A, B and C entered into partnership. A paid in as much as B and one third of C; B paid in as much as C and one third of A, and C paid in £10 and one third of A. What did each man contribute to the stock?

27. It is required to divide the number 91 into two such parts, that the greater being divided by their difference, the quotient may be 7.

28. Divide the number 49 into two such parts, that the greater increased by 6 may be to the less diminished by 11 as 9 to 2.

29. Three merchants A, B and C enter into a speculation; B subscribes £10 more than four-fifths of what A does, and C £30 more than half of what B does; A's gain is two-fifths of his subscription and B's is £148. What are the respective sums subscribed and whole gain?

Art 21. Thus far, the operations required in the solution of a question have each been performed, as soon as determined. No trace of the operations therefore appears in the result. Let us now resume the question art. 1, and instead of performing the operations, as we proceed, let us retain them by means of the proper signs. Representing as before the lesser part by x, the greater will be $x + 4$, and we have

$$x + x + 4 = 18$$
$$2x + 4 = 18$$
$$2x = 18 - 4$$
$$x = \frac{18}{2} - \frac{4}{2}$$

Here the operations, which pertain to the solution of the proposed question appear in the result, according to which to find the lesser part required, from one half of 18 we subtract one half of 4.

22. If the reasoning pursued in the solution of the proposed be examined with attention, it will be perceived, that it does not at all depend upon the particular numbers 18 and 4 given in the question. It will be precisely the same for any other numbers. The same operations will therefore be necessary to obtain the lesser part, whatever the numbers given in the question may be.

By preserving the operations therefore, we obtain in the result a *general solution* of the proposed, that is, we determine at once what operations are necessary for all questions, which differ from the proposed only in the particular numbers, which are given.

Art 23. To resolve a question in a general manner it will be more convenient, however, to represent the given things in the question by signs, which may stand indifferently for any numbers whatever.

It is agreed to represent known quantities, or those which are supposed to be given in a question, by the first letters of the alphabet, as a, b, c.

Representing by a the number to be divided, and by b the given excess, the question under consideration may be presented generally, thus ; To divide a number a into two such parts, that the greater may exceed the less by b. To resolve the question, thus stated, we denote still the lesser part by x ; the greater will then be $x + b$, and we have

$$x + x + b = a$$
$$2x + b = a$$
$$2x = a - b$$
$$x = \frac{a}{2} - \frac{b}{2}$$

24. The above expression for x is called a *formula* for x, since it indicates the operations to be performed upon the numbers represented by a and b in order to obtain x.

The translation of a formula into common language is called a *rule*. Thus we have the following rule, by which to obtain the lesser of the parts required according to the question proposed, viz. *From half the number to be divided, subtract half the given excess, the remainder will be the answer.*

25. Knowing the lesser part, we obtain the greater by adding to the less the given excess. We may, however, easily obtain a rule for calculating the greater part without the aid of the less. Indeed since the lesser part is equal to $\frac{a}{2} - \frac{b}{2}$, if we add b to this, we have $\frac{a}{2} - \frac{b}{2} + b$ equal to the

greater. But this expression may, it is easy to see, be reduced to $\frac{a}{2} + \frac{b}{2}$; whence we have the following rule, by which to find the greater part, viz. *To half the number to be divided, add half the given excess, the result will be the answer.*

26. The third question art. 7 may be presented in a general manner, thus; To divide a number a into three such parts, that the excess of the mean above the least may be b, and the excess of the greatest above the mean may be c.

Let $x =$ the least part;
then $x + b =$ the mean,
and $x + b + c =$ the greatest

therefore $x + x + b + x + b + c = a$

or transposing and reducing $3x = a - 2b - c$

whence $x = \frac{a - 2b - c}{3}$

Translating the above formula into common language, we have the following rule, by which to find the least part, viz. *From the number to be divided, subtract twice the excess of the mean part above the least, and also the excess of the greatest above the mean, and take a third of the remainder.*

To apply this rule, let it be required to divide \$973 among three men, so that the second shall have \$69 more than the first, and the third \$43 more than the second.

The fourth, fifth, and sixth questions art. 7 may also, it will easily be perceived, be solved by means of the same rule.

27. To obtain a formula for the mean part, we add b, the excess of the mean above the least, to the above expression for the least part, which gives for the mean

$$\frac{a - 2b - c}{3} + b$$

or reducing to a common denominator

$$\frac{a - 2b - c}{3} + \frac{3b}{3}$$

whence we obtain for the mean part

$$\frac{a + b - c}{3}$$

In like manner the following formula will readily be obtained for the greatest part, viz. $\frac{a+b+2c}{3}$

28. The first question art. 19 may be stated generally, thus. A cistern is supplied by two pipes; the first will fill it in a hours, the second in b hours. In what time will the cistern be filled if both run together?

Let $x =$ the time; the capacity of the cistern being supposed equal to unity, we have

$$\frac{x}{a} + \frac{x}{b} = 1$$

whence freeing from denominators

$$a x + b x = a b$$

Here it will be observed, that x is taken a times and also b times; whence on the whole it is taken $a + b$ times; $a + b$ is then the coefficient of x, and the above equation may be written, thus,

$$(a + b) x = a b$$

whence $$x = \frac{a b}{a + b}$$

Translating this formula, we have the following rule for every case of the proposed question, viz. *Divide the product of the numbers, which denote the times employed by each pipe in filling the cistern, by the sum of these numbers; the quotient will be the time required by both the pipes running together to fill the cistern.*

Example. Suppose one pipe will fill the cistern in $7\frac{1}{2}$ hours, and the other in 9 hours; in what time will it be filled if both run together?

29. The fourth question art. 19 may be presented in a general manner thus. A laborer was hired for a certain number a of days; for each day that he wrought he was to receive b shillings, but for each day that he was idle, he was to forfeit c shillings. At the end of the time he received d shillings. How many days did he work, and how many was he idle?

Putting $x =$ the number of days, in which he wrought, and resolving the question, we obtain

$$x = \frac{d + a c}{b + c}$$

EXAMPLE. A laborer was hired for 75 days; for each day that he wrought he was to receive \$3, but for each day that he was idle, he was to forfeit \$7. At the end of the time he received \$125. To determine by the above formula the number of days, in which the laborer wrought.

The following questions may also, it is easy to see, be solved by the same formula.

A man agreed to carry 20 earthen vessels to a certain place on this condition; that for every one delivered safe he should receive 11 cents, and for every one he broke, he should forfeit 13 cents; he received 124 cents. How many did he break?

A fisherman to encourage his son promises him 9 cents for each throw of the net in which he should take any fish, but the son, on the other hand, is to forfeit 5 cents for each unsuccessful throw. After 37 throws the son receives from the father 235 cents. What was the number of successful throws of the net?

30. Three men A, B, and C commence trade together, and furnish money in proportion to the numbers m, n, and p respectively; they gain a certain sum a. What is each man's share of the gain?

Let $x =$ A's share;

then $\frac{nx}{m} =$ B's, and $\frac{px}{m} =$ C's share.

By the question therefore

$$x + \frac{nx}{m} + \frac{px}{m} = a$$

Freeing from denominators, we have

$$m x + n x + p x = m a$$

or, which is the same thing

$$(m + n + p)\, x = m a$$

whence $$x = \frac{m a}{m + n + p} = \text{A's share.}$$

Multiplying next the value of x by n, and dividing by m we obtain

$$\frac{na}{m+n+p} = \text{B's share.}$$

In like manner, we find

$$\frac{pa}{m+n+p} = \text{C's share}$$

To find a share of the gain therefore; *Multiply the corresponding proportion of the stock into the whole gain, and divide by the sum of the proportions.*

This is the rule of *Fellowship* given in arithmetic.

31. Thus far we have employed the first letters of the alphabet to represent known quantities, and the last to denote those, which are unknown. In some cases it is more convenient to represent the quantities, whether known or unknown, by the initials of the words for which they stand.

32. Let it be proposed to determine what sum of money must be put at interest, at a given rate per cent, in order to amount to a given sum in a given time.

Let $p =$ the principal or sum put at interest,
$r =$ the rate per cent,
$a =$ the given amount,
$t =$ the given time.

By the question, we have $p + trp = a$

or $(1 + tr)p = a$

whence $p = \dfrac{a}{1+tr}$

We have therefore the following rule, by which to find the principal required, viz. *Multiply the rate by the time and add 1 to the product; the amount divided by the sum thus obtained will give the principal.*

Examples. 1. What sum of money must be put at interest at 6 per cent, in order that the principal and interest may, at the end of 5 years, amount to \$748,80?

2. A man lent a certain sum of money at 5 per cent; at the end of 7 years he received for principal and interest \$1237,47. What was the sum lent?

Art – 33. The equation $p + trp = a$, contains, it will be perceived, four different things, any one of which may be determined, when the others are known. Deducing, for example, the value of t, we have

$$t = \frac{a - p}{rp}$$

Whence to find the time, when the amount, principal, and rate are given; *From the amount subtract the principal, and divide the remainder by the product of the rate multiplied by the principal.*

EXAMPLE. A man put at interest \$345 at 4 per cent; at the end of a certain time he received for principal and interest \$483. Required the time for which the money was lent.

Let the learner investigate a rule, by which to find the rate, when the amount, principal, and time are given.

34. In the preceding questions the object has been to determine certain unknown numbers by means of others, which are known, and which have with the first relations established by the enunciation of the question. We shall now show the aid derived from the same signs in demonstrating certain properties in relation to known and given numbers.

1°. To demonstrate that if both terms of a fraction be multiplied by the same number, the value of the fraction will not be changed.

Let the proposed fraction be designated by $\frac{a}{b}$, and let n be any number whatever.

Putting $\frac{a}{b} = m$, we have $a = bm$

multiplying both sides of this last by n, we have

$$na = nbm$$

from which we deduce

$$\frac{na}{nb} = m$$

whence $$\frac{na}{nb} = \frac{a}{b}$$

2°. Let the same number be added to both terms of a proper fraction; to determine what effect this will produce upon the value of the fraction.

Let us designate the fraction by $\frac{a}{b}$. Let m be the number added to both terms of this fraction; it then becomes

$$\frac{a+m}{b+m}$$

To compare the two fractions, it is necessary to reduce them to the same denominator. Performing this operation, we have for the first

$$\frac{ab+am}{bb+bm}$$

and for the second

$$\frac{ab+bm}{bb+bm}$$

Here the two numerators have the part ab common to both; but the part bm of the second is greater than the part am of the first, since b is greater than a; the second fraction is therefore greater than the first; Whence, *If the same number be added to both terms of a proper fraction, the value of the fraction will be increased.*

3°. It has been shown in arithmetic that, *Every divisor common to two numbers must divide the remainder after their division.* Let us now demonstrate this property by the aid of algebraic symbols.

Let D be the divisor common to the two numbers; let A D represent the greater of the two numbers and B D the lesser; let Q be the entire quotient arising from their division, and R the remainder; we have then

$$AD = BD \times Q + R$$

dividing both sides by D, we have

$$A = B \times Q + \frac{R}{D}$$

Here the first member of the equation is an entire number, the second must therefore be equal to an entire number; but of this member the term B Q is an entire number; whence $\frac{R}{D}$ must be an entire number, that is R must be exactly divisible by D. The proposition above is therefore demonstrated.

The questions art. 20 will furnish exercises for the learner in stating and resolving questions in a general manner.

SECTION III. ALGEBRAIC OPERATIONS.

Art 35. A quantity expressed by algebraic signs is called an *algebraic* or *literal* quantity.

From what has been done, it is easy to see that we shall have frequent occasion to perform upon algebraic quantities operations analogous to the fundamental operations of arithmetic, viz. addition, subtraction, multiplication and division. It is to be observed, however, that the operations upon algebraic quantities differ from the corresponding ones in arithmetic in this respect, that the results to which we arrive in the case of algebraic quantities are for the most part only indications of operations to be performed. All that we do is to transform the operations originally indicated into others, which are more simple, or which become necessary in order that the conditions of the question may be fulfilled. Thus, in the equation $x + x + b = a$ given by the conditions of the question art. 23, we simplify the operations originally indicated by reducing the expression $x + x$ to one term, $2x$, by an operation analogous to addition in arithmetic, though not strictly the same. So likewise in question fourth art. 19, though we cannot strictly speaking subtract $576 - 12x$ from $24x$, yet by an operation analogous to subtraction in arithmetic, we indicate upon [illegible] quantities operations, which produce the same effect, as the subtraction which the conditions of the question require.

36. Algebraic quantities consisting only of one term are called *simple* quantities, as $3a$, $-4b$ &c. Those which consist of two terms are called *binomials*, as $a + b$, $c - d$. Those which consist of three terms are called *trinomials* &c. In general, quantities consisting of more than one term are called *polynomials*, sometimes also *compound quantities*.

Quantities in algebra, which are composed of the same letters, and in which the same letters are repeated the same

number of times are called similar quantities, thus, $3\,a\,b$, $7\,a\,b$ are similar quantities, so also $a\,a\,b$, $5\,a\,a\,b$.

ADDITION OF ALGEBRAIC QUANTITIES.

37. 1°. Let it be required to add the simple quantities a, b, c, and d; the result, it is evident, will be $a + b + c + d$.

2°. Let the quantities to be added be $a\,b$, c, $a\,b$, d. Here we have as before $a\,b + c + a\,b + d$; but the quantities $a\,b$, $a\,b$ in this result are similar, they may therefore be united in one term, thus, $2\,a\,b$; whence the sum required will be $2\,a\,b + c + d$. To add simple quantities therefore, *Write them one after the other with the sign* $+$ *between them, observing to simplify the result by uniting in one, terms which are similar.*

3°. Let it next be required to add the polynomials $a + b$ and $c + d + e$. The sum total of any number of quantities whatever should be equal, it is evident, to the sum of all the parts of which these quantities are composed; we have therefore for the sum required $a + b + c + d + e$.

Let the quantities proposed be $a + b$ and $c - d$. If we begin by adding c, the result $a + b + c$ will, it is evident, be too great by the quantity d by which c is diminished; to obtain the true result therefore, from $a + b + c$ we must subtract d; whence $c - d$ added to $a + b$ gives

$$a + b + c - d.$$

To add polynomials therefore, *Write in order one after the other the quantities to be added with their proper signs, it being observed that the terms, which have no signs before them, are considered as having the sign* $+$.

38. Let it next be required to add the following quantities,

$$\begin{array}{l} 9\,a + 7\,b - 2\,c \\ 2\,a - 5\,c \\ 8\,b + c \end{array}$$

By the rule just given the sum required will be

$$9\,a + 7\,b - 2\,c + 2\,a - 5\,c + 8\,b + c$$

In this result the similar terms $9\,a$, $2\,a$ may be united in one sum $11\,a$; also the terms $7\,b$ and $8\,b$ give $15\,b$.

The similar quantities $-2c$, $-5c$ being both subtractive, the effect will be the same, if we unite them in one sum $7c$ and subtract this sum, and as there would still remain the quantity c to be added, instead of first subtracting $7c$ and then adding c to the result, the effect will be the same if we subtract only $6c$.

The sum of the expressions proposed will then be reduced to $11a + 15b - 6c$.

The operation, by which all similar terms are reduced to one, whatever signs they may have, is called *reduction*. To perform this operation, *Take the sum of similar quantities, which have the sign + and that of those, which have the sign —; subtract the less of the two sums from the greater and give to the remainder the sign of the greater.*

Art. 39. We have then the following general rule for the addition of algebraic quantities, viz. *Write the quantities in order one after the other with their proper signs, observing to simplify the result by reducing to one, terms which are similar.*

EXAMPLES.

1. To add the quantities

$$\begin{array}{ll}
 & 5a + 3b - 4c \\
 & 6c + 2a - 5b + 2d \\
 & 3e - 4b - 2c + a \\
 & 7a - 3c + 4b - 6e \\
\hline
\text{Answer} & 15a - 2b - 3c + 2d - 3e
\end{array}$$

2. To add the quantities

$$\begin{array}{ll}
 & 7m + 3n - 14p + 17r \\
 & 3a + 9n - 11m + 2r \\
 & 5p - 4m + 8n \\
 & 11n - 2b - m - r + s \\
\hline
\text{Answer} & 31n - 9m - 9p + 18r + 3a - 2b + s.
\end{array}$$

3. To add the quantities

$$\begin{array}{ll}
 & 11bc + 4ad - 8ac + 5cd \\
 & 8ac + 7bc - 2ad + 4mn \\
 & 2cd - 3ab + 5ac + am \\
 & 9am - 2bc - 2ad + 5cd \\
\hline
\text{Answer} & 16bc + 5ac + 12cd + 4mn - 3ab + 10am.
\end{array}$$

SUBTRACTION OF ALGEBRAIC QUANTITIES.

40. 1. To subtract a from b. Here the quantities being dissimilar, the subtraction can only be expressed by the sign —, thus, $b - a$.

2. To subtract $5a$ from $7a$. The quantities in this case being similar, the subtraction may be performed by means of the coefficients, and the result, it is evident, will be $2a$.

3. To subtract $2b + 3c$ from d. To subtract one quantity from another we must, it is evident, take from this other the sum of all the parts, of which the quantity to be subtracted is composed. The result required will therefore be $d - 2b - 3c$.

4. To subtract $a - b$ from c. If we begin by subtracting a from c, it is evident, that we shall take away too much by the quantity b, by which a should be diminished before its subtraction; b should therefore be added to $c - a$ to give the true result; whence $a - b$ subtracted from c gives $c - a + b$.

5. To subtract $5c + 3d - 4b$ from $7c - 2d - 5b$. The result, it is easy to see, will be

$$7c - 2d - 5b - 5c - 3d + 4b$$

which becomes by reduction

$$2c - 5d - b$$

41. From what has been done the following rule for the subtraction of algebraic quantities will be readily inferred, viz. *Change the signs + into —, and the signs — into + in the quantities to be subtracted, or suppose them to be changed, and then proceed as in addition.*

EXAMPLES.

1. To subtract from	$17a + 2m - 9b - 4c + 23d$
the quantity	$51a - 27b + 11c - 4d$
Answer	$2m - 34a + 18b - 15c + 27d$

2. To subtract from	$5ac - 8ab + 9bc - 4am$
the quantity	$8am - 2ab + 11ac - 7cd$
Answer	$9bc - 6ac - 6ab - 12am + 7cd$

Art 41 3. To subtract from $15abc - 13xy + 21cd - 41x - 25$
the quantity $75xy - 4abc + 16x - 53cd - 31mc$
Answer $19abc - 88xy - 57x + 74cd + 31mc - 25$

MULTIPLICATION OF ALGEBRAIC QUANTITIES.

Artic 42. 1. The product of a quantity a by another quantity b is expressed, as we have already seen, thus, $a \times b$, or in a more simple manner, thus, ab. In like manner the product of ab by cd is expressed thus, $ab \times cd$, or thus, $abcd$.

2. The letters a and b are called *factors* of the product ab. So also a, b, c and d are the factors of the product $abcd$. The value of a product, it is easy to see, does not depend at all upon the order, in which its factors are arranged; thus the value of the product arising from the multiplication of a by b will evidently be the same, whether we write ba or ab.

3. Let it be proposed to multiply $3ab$ by $5cd$; by no. 1 we have $3ab\,5cd$, or by no. 2, $3 \times 5abcd$; but the factors 3 and 5 in this result may, it is evident, be reduced to one by multiplying them together; performing this operation, the product required will be $15abcd$. In like manner the product of the quantities $7ab$, $9cd$, $13ef$ will be $819abcdef$.

4. Let it be required to multiply aa by a. According to no. 1 we have for the result aaa; but this expression for the product required may, it is easy to see, be abridged by writing the letter a but once only, and indicating by a figure the number of times this letter enters into it as a factor. The figure which indicates the number of times a given letter enters as a factor in a product is called the *exponent* of that letter. And in order to distinguish the exponent of a letter from a coefficient, we place the exponent at the right hand of the letter and a little above it, the coefficient being always placed before the letter, to which it belongs, and on the same line with it.

According to this method the product aa is expressed by a^2, aaa by a^3, $aaaa$ by a^4, &c.

A letter, which is multiplied once by itself, or which has two for an exponent, is said to be raised to the *second power.* A letter which is multiplied twice successively by itself, or which has 3 for an exponent, is said to be raised to the *third* power. In general, the power of a letter is designated according to the figure, which it has for an exponent, thus a with 7 for an exponent is called the *seventh* power of a.

A letter which has no exponent is considered as having unity for its exponent, thus, a is the same as a^1.

From what has been said, it will be perceived, that *in order to raise a letter to a given power, it is necessary to multiply it successively by itself as many times less one, as there are units in the exponent of this power.*

5. Let it next be required to multiply a^3 by a^5. According to no. 1 the product will be expressed by $a^3 a^5$. In this product the letter a, it will be observed, occurs three times as a factor and also five times as a factor, whence on the whole it is found eight times as a factor. The product $a^3 a^5$ may therefore according to no. 4 be expressed more concisely, thus, a^8. In like manner the product of a^7 by a^9 will be a^{16}. Whence, in general, *The product of two powers of the same letter will have for an exponent the sum of the exponents of the multiplier and multiplicand.*

6. Let it be proposed next to multiply $a^5 b^2 c$ by $a^4 b^3 c^2 d$. According to no. 1 the product will be $a^5 b^2 c a^4 b^3 c^2 d$, or by no. 2, $a^5 a^4 b^2 b^3 c c^2 d$; but this expression may be reduced by the rule just given to $a^9 b^5 c^3 d$; whence

$$a^5 b^2 c \times a^4 b^3 c^2 d = a^9 b^5 c^3 d.$$

43. From what has been done we have the following rule for the multiplication of simple quantities, viz. 1°. *Multiply the coefficients together* ; 2°. *write in order in the product thus obtained the letters which are found at once in both the multiplier and multiplicand, observing to give to each letter the sum of the exponents, with which this letter is affected in the two factors* ; 3°. *if a letter is found in one of the factors only, write it in the product with the exponent which it has in this factor.*

EXAMPLES.

To multiply 1. $8\,a^2\,b\,c^2$ by $7\,a\,b\,c\,d^2$ Ans. $56\,a^3\,b^2\,c^3\,d^2$.
2. $21\,a^3\,b^2\,c\,d$ by $8\,a\,b\,c^3$ Ans. $168\,a^4\,b^3\,c^4\,d$.
3. $17\,a\,b^6\,c$ by $7\,d\,f$ Ans. $119\,a\,b^6\,c\,d\,f$.

44. Let us pass to the multiplication of polynomials.

To indicate that a polynomial $a+b$, for example, is multiplied by another $c+d$, we draw a vinculum over each and connect them by the sign of multiplication, thus,

$$\overline{a+b}\times\overline{c+d},$$

or which is the better method, we inclose each of the quantities in a parenthesis and write them in order one after the other, either with or without the sign of multiplication, thus,

$$(a+b)\times(c+d),\text{ or }(a+b)\,(c+d).$$

1. To multiply $a+b$ by c. To form the product required, it is evident, that we must take c times each of the parts a and b of which the quantity $a+b$ is composed.

The product of	$a+b$
multiplied by	c
is therefore	$ac+bc$
In like manner	$2\,a+b^2\,c+d$
multiplied by	h
gives	$2\,ah+b^2\,ch+dh$

2. To multiply $a-b$ by c. Since $a-b$ is smaller than a by the quantity b, ac the product of a by c, it is evident, will be too large for the product required by b times c or bc; whence to obtain the true result, from ac we must subtract bc.

The product of	$a-b$
multiplied by	c
is therefore	$ac-bc$
In like manner	$a^2+c^2-dh-ef$
multiplied by	ah
gives	$a^3\,h+ahc^2-adh^2-ahef$

From what has been done it is evident that, *If two terms each affected with the sign + be multiplied together, the product must have the sign +; but if one of the terms be affected with the sign + and the other with the sign —, the product should have the sign —.*

3. Let it be proposed next to multiply $a - b$ by $c - d$. In this case, it is evident, that, if we take c times $a - b$ the result will be too great by d times $a - b$; whence to obtain the true product, from c times $a - b$ or $ac - bc$ we must subtract d times $a - b$ or $ad - bd$,

$$\begin{array}{ll} \text{The product of} & a - b \\ \text{multiplied by} & c - d \\ \hline \text{is therefore} & ac - bc - ad + bd \end{array}$$

From this example it appears that, *If two terms be affected each with the sign —, the product of these terms should be affected with the sign +.*

If in the expression of a product there occur similar terms, the expression may be abridged by uniting these terms into one.

$$\begin{array}{ll} \text{Thus} & 2ab^2 + a^2 - c^2 \\ \text{multiplied by} & a^2 - ab^2 + c^2 \\ \hline & 2a^3b^2 + a^4 - a^2c^2 \\ & -2a^2b^4 - a^3b^2 + ab^2c^2 \\ & 2ab^2c^2 + a^2c^2 - c^4 \\ \hline \text{gives} & a^3b^2 + a^4 - 2a^2b^4 + 3ab^2c^2 - c^4 \end{array}$$

45. From what has been done we have the following rule for the multiplication of polynomials, viz. 1°. *Multiply each term of the multiplicand by each term of the multiplier, observing with respect to the signs, that if two terms multiplied together have each the same sign, the product should have the sign +, but if they have different signs, the product must have the sign —; 2°. add together the partial products thus obtained, taking care to unite in one, terms which are similar.*

A polynomial is said to be *arranged* with reference to

some letter, when its terms are written in order according to the powers of this letter. The polynomial

$$a^2 b^2 + a^3 b - a b^4 + a^4,$$

for example, arranged according to the descending powers of the letter a stands thus, $a^4 + a^3 b + a^2 b^2 - a b^4$.

To facilitate the multiplication of polynomials, it is usual, 1°. to arrange the quantities to be multiplied according to the powers of the same letter; 2°. to dispose of the partial products in such a manner that those terms, which are similar, shall fall under each other. Let it be proposed, for example, to multiply

$$b^3 + b^2 a + a^3 + b a^2 \text{ by } 4 b^2 - 3 b a + 3 a^2.$$

The multiplier and multiplicand being both arranged with reference to the letter a, the work will be as follows.

$$\begin{array}{l}
a^3 + b a^2 + b^2 a + b^3 \\
3 a^2 - 3 b a + 4 b^2 \\
\hline
3 a^5 + 3 b a^4 + 3 b^2 a^3 + 3 b^3 a^2 \\
\qquad - 3 b a^4 - 3 b^2 a^3 - 3 b^3 a^2 - 3 b^4 a \\
\qquad\qquad\qquad 4 b^2 a^3 + 4 b^3 a^2 + 4 b^4 a + 4 b^5 \\
\hline
3 a^5 + 4 b^2 a^3 + 4 b^3 a^2 + b^4 a + 4 b^5
\end{array}$$

46. The following examples will serve as an exercise in the multiplication of polynomials.

To multiply

1. $5 a^3 - 4 a^2 b + 5 a b^2 - 3 b^3$

by $4 a^2 - 5 a b + 2 b^2$

Answer $20 a^5 - 41 a^4 b + 50 a^3 b^2 - 45 a^2 b^3 + 25 a b^4 - 6 b^5$

2. $a^3 + 3 a^2 b + 3 a b^2 + b^3$

by $a^3 - 3 a^2 b + 3 a b^2 - b^3$

Answer $a^6 - 3 a^4 b^2 + 3 a^2 b^4 - b^6$

3. $x^4 + x^3 y + x^2 y^2 + x y^3 + y^4$

by $x - y$

Answer $x^5 - y^5$

47. A term, which contains one literal factor only, is said to be of the *first degree*; a term, which contains two literal factors only, is said to be of the *second* degree, &c. In general, *the degree of a term is marked by the number, which expresses the sum of the exponents of the letters, which enter into this term.* The coefficient is not reckoned in estimating the degree of the term. Thus $a^2 b^3 c$ is a term of the 6th degree, and $7\,a b^3$ is a term of the fourth degree.

A polynomial is said to be *homogeneous*, when all its terms are of the same degree. Thus, $3\,a^2 - 4\,a\,b$, $5\,a^3 + a\,b\,c - b^3$ are homogeneous polynomials.

48. From the rules for multiplication, which have been laid down, it follows,

1°. If the polynomials proposed for multiplication are each homogeneous, *the product of these polynomials will also be homogeneous*, and *the degree of each term of the product should be equal to the sum of the degrees of any two terms whatever of the multiplier and multiplicand.* Thus, in the first example art. 46, all the terms of the multiplicand being of the third degree and those of the multiplier of the second degree, all the terms of the product are of the fifth degree. When therefore the factors of a product are homogeneous, we may readily detect by means of this remark any error in regard to the exponents, which may have occurred in the course of the work.

2°. In the multiplication of polynomials, if there be no reduction of similar terms, *the number of terms in the product will be equal to the number of terms in the multiplicand multiplied by the number of terms in the multiplier.* Thus, if there be 5 terms in the multiplicand and 4 in the multiplier, there will be 20 in the product.

3°. But if there be a reduction of similar terms, then the number of terms in the product may be much less. It is to be observed, however, that among the different terms of the product there will be two at least, which will not admit of reduction with any other, viz. 1°. *The term arising from the multiplication of the term in the multiplicand affected with the highest exponent of one of the letters, by the term in the multiplier affect-*

ed with the highest exponent of the same letter. 2°. *the term arising from the multiplication of the two terms affected with the lowest exponent of the same letter.*

49. The following examples in the multiplication of polynomials deserve attention, on account of the important conclusions to be derived from them.

1. To multiply $a + b$ by $a + b$, we have

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

From this result we learn that, *The second power or square of the sum of two quantities contains the square of the first quantity, plus double the product of the first by the second, plus the square of the second.*

Thus, $(7 + 3)(7 + 3)$ or, $(7 + 3)^2 = 49 + 42 + 9 = 100$

So also $(5a^3 + 8a^2b)^2 = 25a^6 + 80a^5b + 64a^4b^2$

2. To multiply $a - b$ by $a - b$, we have

$$(a - b)(a - b) = a^2 - 2ab + b^2$$

From this result we learn, that *the square of the difference of two quantities is composed of the square of the first quantity, minus double the product of the first by the second, plus the square of the second.*

Thus, $(7a^2b^2 - 12ab^2)^2 = 49a^4b^4 - 168a^3b^4 + 144a^2b^4$

3. To multiply $a + b$ by $a - b$, we have

$$(a + b)(a - b) = a^2 - b^2$$

Whence, *the sum of two quantities multiplied by their difference is equal to the difference of their squares.*

Thus, $(7 + 4)(7 - 4) = 49 - 16 = 33$

So also $(8a^3 + 7ab^2)(8a^3 - 7ab^2) = 64a^6 - 49a^2b^4$

The manner in which an algebraic product is formed by means of its factors is called the *law* of this product. This law, it will readily be perceived, remains always the same, whatever may be the values attributed to the letters which enter into the factors.

50. A polynomial being given, we may sometimes by mere inspection decompose it into its factors, an operation which is frequently useful.

Let there be the polynomial $25\,a^4 - 30\,a^3\,b + 15\,a^2\,b^2$; the factors 5 and a^2, it is evident, enter into each of the terms; the proposed therefore may be put under the form

$$5\,a^2\,(5\,a^2 - 6\,a\,b + 3\,b^2)$$

In like manner the polynomial $64\,a^4\,b^6 - 25\,a^2\,b^8$ may be put under the form

$$(8\,a^2\,b^3 + 5\,a\,b^4)(8\,a^2\,b^3 - 5\,a\,b^4)$$

for, the proposed, being the difference of the squares of the two quantities $8\,a^2\,b^3$, $5\,a\,b^4$, will be equal to the product of their sum and difference.

DIVISION OF ALGEBRAIC QUANTITIES.

51. 1. The object of division in algebra is the same as that of division in arithmetic, viz. *to find one of the factors of a given product, when the other is known.*

According to this definition the divisor multiplied by the quotient must produce anew the dividend; the dividend therefore must contain all the factors both of the divisor and quotient; *Whence the quotient is obtained by striking out of the dividend the factors of the divisor.*

Thus to divide $a\,b\,c\,d$ by $a\,c$, we strike out of the dividend the factors a and c of the divisor and obtain $b\,d$ for the quotient.

2. Let it be required to divide $a^5\,b$ by $a^2\,b$. Decomposing a^5 into the two factors a^3 and a^2, the dividend may be put under the form $a^3\,a^2\,b$; whence striking out of the dividend the factors a^2 and b of the divisor, the quotient will be a^3.

From this example it appears that in order to find the quotient of two powers of the same letter; *From the exponent of the dividend we subtract that of the divisor, the remainder will be the exponent of the quotient.*

3. If it be required to divide $72\ a\ b^7\ c$ by $9\ b^2$, we find that 72, the coefficient of the dividend, may be decomposed into the two factors 9 and 8; b^7 may also be decomposed into the two factors b^5 and b^2; the dividend therefore may be put under the form $9 \times 8\ a\ b^5\ b^2\ c$; whence, suppressing 9 and b^2, the factors of the divisor, we have $8\ a\ b^5\ c$ for the quotient.

52. From what has been said we have the following rule for the division of simple quantities, viz. 1°. *divide the coefficient of the dividend by the coefficient of the divisor; 2°. suppress in the dividend the letters, which are common to it and the divisor, when they have the same exponent, and when the exponent is not the same, subtract the exponent of the divisor from that of the dividend and the remainder will be the exponent to be affixed to the letter in the quotient; 3°. write in the quotient the letters of the dividend, which are not in the divisor.*

EXAMPLES.

1. To divide $48\ a^3\ b^3\ c^2\ d$ by $12\ a\ b^2\ c$ Ans. $4\ a^2\ b\ c\ d$.

2. To divide $150\ a^5\ b^8\ c\ d^3$ by $30\ a^3\ b^5\ d^2$ Ans. $5\ a^2\ b^3\ c\ d$.

53. From the preceding rule, it is evident, in order that the division may be possible, 1°. that the coefficient of the divisor should exactly divide the coefficient of the dividend; 2°. the exponent of a letter in the divisor should not exceed the exponent of the same letter in the dividend; 3°. that there should be no letter in the divisor, which is not found in the dividend.

When these conditions do not exist, the division can only be indicated by the usual sign. If it be required, for example, to divide $12\ a^2\ b$ by $9\ c\ d$, the division it is easy to see, cannot be performed, we therefore express the quotient by writing the divisor under the dividend in the form of a fraction, thus, $\frac{12\ a^2\ b}{9\ c\ d}$.

54. The expression $\frac{12\ a^2\ b}{9\ c\ d}$ is called an *algebraic fraction.*

Fractions of this species may be simplified, in the same manner as those of arithmetic, by striking out the factors, which are common to both terms, or which is the same thing, by dividing both terms by the factors, which are common to them.

Let it be required, for example, to divide $48\,a^3\,b^2\,c\,d^3$ by $36\,a^2\,b^3\,c^2\,d\,e$; from what has been said, the most simple expression for the quotient will be $\frac{4\,a\,d^2}{3\,b\,c\,e}$

In like manner $a^2\,b$ divided by $5\,a^3\,b$ gives $\frac{1}{5\,a}$ for the quotient.

55. It sometimes happens, that the exponent of a letter is the same both in the divisor and dividend. The rule for obtaining the exponents of the letters of the quotient art. 51, being applied to a case of this kind, will give zero for the exponent of the letter in the quotient. Thus, $\frac{a^2}{a^2}$ according to this rule gives a^0 for a quotient; but $\frac{a^2}{a^2}$, it is evident, is equal to unity, the expression a^0 may therefore be considered as equivalent to unity. In general, *a letter with zero for an exponent is to be regarded as a symbol equivalent to unity.*

This symbol, it is evident, will produce no effect upon the value of the expression, in which it appears as a factor, since it signifies nothing but unity. Its only use is to preserve in the work the trace of a letter, which formed a part of the question proposed, but which would otherwise disappear by the effect of division. Thus, if it be required to divide $24\,a^3\,b^2$ by $8\,a^2\,b^2$, the quotient from what has been said may be put under the form $3\,a\,b^0$. The symbol b^0 indicates that the letter b enters o times as a factor in this result, or in other words that it does not enter into it as a factor, but at the same time it serves to show that this letter belonged as a factor to the quantities, from which the result $3\,a$ is obtained by division.

56. We pass next to the division of polynomials. Since the divisor multiplied by the quotient should produce anew the dividend, it is evident, that the dividend must contain all the partial products arising from the multiplication of each term of the divisor by each term of the quotient. This being the case, it is easy to see, that if we can find any one of these partial products in the dividend, and the particular term of the divisor upon which it depends is known, by dividing this term in the dividend by the known term of the divisor, we shall obtain a term of the quotient sought.

Let it be required to divide

$$50\,a^3\,b^2 - 41\,a^4\,b + 20\,a^5 + 10\,a\,b^4 - 33\,a^2\,b^3$$

by $\qquad 5\,a\,b^2 - 4\,a^2\,b + 5\,a^3.$

It is evident from what has been said art. 48, that the term a^5, being affected with the highest exponent of the letter a in the dividend, must have been formed without any reduction from the multiplication of $5\,a^3$, the term affected with the highest exponent of the letter a in the divisor, by the term affected with the highest exponent of the same letter in the quotient, that is, the term $20\,a^5$ of the dividend is the product of $5\,a^3$ of the divisor by a term of the quotient; whence, dividing $20\,a^5$ by $5\,a^3$, we obtain $4\,a^2$ one of the terms of the quotient sought. Multiplying the divisor by $4\,a^2$, we produce anew all the terms of the dividend, which depend upon $4\,a^2$, viz. $20\,a^3\,b^2 - 16\,a^4\,b + 20\,a^5$; subtracting these from the dividend, the remainder

$$30\,a^3\,b^2 - 25\,a^4\,b + 10\,a\,b^4 - 33\,a^2\,b^3$$

must contain all the partial products arising from the multiplication of each one of the remaining terms of the quotient by each term of the divisor.

Regarding this remainder as a new dividend, it is evident, from what has been said, that the term $-\,25\,a^4\,b$ must have arisen from the multiplication of $5\,a^3$ by the term affected with the highest exponent of the letter a in the remaining terms of the quotient sought; whence, dividing

$-25\,a^4 b$ by $5\,a^3$, we shall be sure to obtain a new term of the quotient.

With regard to the sign, which should be prefixed to this term of the quotient, it is evident, that it should be the sign $-$; since, from the nature of multiplication, the divisor having the sign $+$, the quotient should have the sign $-$ in order that their product may produce anew the dividend $-25\,a^4 b$.

Performing the operation therefore, we have $-5\,a\,b$ for another term of the quotient sought. Multiplying the divisor by this term of the quotient, we obtain all the terms of the dividend, which depend upon $-5\,a\,b$, viz.

$$-25\,a^2 b^3 + 20\,a^3 b^2 - 25\,a^4 b,$$

subtracting these from $30\,a^3 b^2 - 25\,a^4 b + 10\,a\,b^4 - 33\,a^2 b^3$, the remainder $10\,a^3 b^2 + 10\,a\,b^4 - 8\,a^2 b^3$, will contain all the partial products arising from the multiplication of each one of the remaining terms of the quotient sought by each term of the divisor; whence, for the same reasons as before, dividing $10\,a^3 b^2$ by $5\,a^3$, we have $2\,b^2$ for a new term of the quotient; multiplying the divisor by this term and subtracting as before, nothing remains; the division is therefore exact, and we have for the quotient sought $4\,a^2 - 5\,a\,b + 2\,b^2$.

57. In the course of reasoning pursued above, we have been obliged to seek in each of the partial operations the term in the dividend, affected with the highest exponent of one of the letters, in order to divide it by the term of the divisor, affected with the highest exponent of the same letter. We avoid this research by arranging the dividend and divisor with reference to the same letter; for, by means of this preparation, the first term at the left of the dividend and the first term at the left of the divisor will, in each of the partial operations, be the two terms which must be divided, one by the other, in order to obtain a term of the quotient.

The following is a table of the calculations in the preceding example, the dividend and divisor being arranged with reference to the letter a, and placed one by the side of the other as in arithmetic.

$$\begin{array}{l|l}
20\,a^5 - 41\,a^4b + 50\,a^3b^2 - 33\,a^2b^3 + 10\,ab^4 & 5a^3 - 4\,a^2b + 5\,ab^2 \\
\underline{20\,a^5 - 16\,a^4b + 20\,a^3b^2} & \overline{4\,a^2 - 5\,a\,b + 2\,b^2} \\
\quad -25\,a^4\,b + 30\,a^3\,b^2 - 33\,a^2\,b^3 + 10\,a\,b^4 & \\
\quad \underline{-25\,a^4\,b + 20\,a^3\,b^2 - 25\,a^2\,b^3} & \\
\qquad\qquad 10\,a^3\,b^2 - 8\,a^2\,b^3 + 10\,a\,b^4 & \\
\qquad\qquad \underline{10\,a^3\,b^2 - 8\,a^2\,b^3 + 10\,a\,b^4} &
\end{array}$$

58. From what has been done, we have the following rule for the division of compound quantities, viz.

Having arranged the divisor and dividend with reference to the powers of the same letter, 1°. *Divide the first term of the dividend by the first term of the divisor, the result will be the first term of the quotient*; 2°. *multiply the whole divisor by the term of the quotient just found, and subtract the result from the dividend*; 3°. *divide the first term of the remainder by the first term of the divisor, the result will be the second term of the quotient*; 4°. *multiply the whole divisor by the second term of the quotient, and subtract the product from the result of the first operation, and continue the same course of operations, until all the terms of the dividend are exhausted.*

Recollecting, that in multiplication the product of two terms affected with the same sign should have the sign +, and that the product of two terms affected with different signs should have the sign —, we infer 1°. *that if the two terms of the dividend and divisor have each the same sign, the quotient arising from their division should have the sign + ; but if they are affected with contrary signs it should have the sign —.* This is the rule for the signs.

EXAMPLES.

To divide

1. $a^5 + 5\,a^4\,b + 7\,a^3\,b^2 + 3\,a^2\,b^3$

by $a^3 + 3\,b\,a^2$

2. $x^5 - x^4\,y - 13\,x^3\,y^2 + x^2\,y^3 + 12\,x\,y^4$

by $x^3 + 2\,x^2\,y - 3\,x\,y^2$

3. $-40\,y^5 + 68\,x\,y^4 + 25\,x^2\,y^3 + 21\,x^3\,y^2 - 18\,x^4\,y - 56\,x^5$

by $5\,y^2 - 6\,x\,y - 8\,x^2$

59. The dividend and divisor being arranged with reference to the powers of the same letter, if the first term of the dividend is not divisible by the first term of the divisor, we infer that the total division is impossible, or in other words, that there is no polynomial, which multiplied by the divisor will reproduce the dividend; and, in general, we infer that the division cannot be exactly performed, *when the first term of any one of the partial dividends is not divisible by the first term of the divisor.*

When the division cannot be exactly performed, in order to complete the quotient, we write the remainder over the divisor in the form of a fraction and annex it to the quotient as in arithmetic.

EXAMPLE.

To divide

$5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 9a^2b^6$

by $5a^4 - 2a^3b + 4a^2b^2$

Answer $a^3 - 4a^2b + 2b^3 + \dfrac{9a^2b^6 - 8a^2b^5}{5a^4 - 2a^3b + 4a^2b^2}$

60. We may remark in passing, that there is some analogy between division in arithmetic and division in algebra with regard to the manner, in which the calculations are disposed and performed; there is however this essential difference, that in arithmetical division the figures of the quotient are obtained by *trial*; whereas, in algebraic division, we obtain with certainty a term of the quotient sought, by dividing the first term of each partial dividend by the first term of the divisor. In algebraic division moreover, we may begin, as it will be easy to see from the remarks art. 48, at the right instead of the left of the dividend; since, in this case, we shall have merely to operate upon the terms affected with the lowest, instead of those affected with the highest exponents of the letter, in reference to which the arrangement is made; whereas, in arithmetical division, we must always begin at the left. Indeed, such is the independence

of the partial operations in algebraic division, that having obtained one of the terms of the quotient and subtracted from the dividend the product of this term by the divisor, we may in the second partial operation divide, one by the other, the two terms of the new dividend and the divisor affected with the highest exponent of any other letter different from that, with reference to which the arrangement is made, and thus obtain a new term of the quotient. It is indeed only for the sake of convenience, that we always regard the same letter in the course of the partial operations necessary to obtain the quotient.

61. In the process of division, the multiplication of the different terms of the quotient by the divisor often produces terms, which are not found in the dividend, and which it is necessary to divide by the first term of the divisor. These terms are such, as cancel each other in the process of forming the dividend by the multiplication of the divisor by the quotient.

To divide $a^3 - b^3$ by $a - b$,

$$\begin{array}{l|l}
a^3 - b^3 & a - b \\ \cline{2-2}
a^3 - a^2 b & a^2 + ab + b^2 \\ \cline{1-1}
\quad a^2 b - b^3 & \\
\quad a^2 b - a b^2 & \\ \cline{1-1}
\qquad a b^2 - b^3 & \\
\qquad a b^2 - b^3 & \\ \cline{1-1}
\end{array}$$

If we now multiply the divisor by the quotient in this example, in order to produce anew the dividend, we shall find, that the new terms, which arise in the process of division, are those which cancel each other in the result of multiplication.

EXAMPLES.

1. To divide $6x^4 - 96$ by $3x - 6$
2. To divide $x^3 - y^3$ by $x + y$
3. To divide $a^4 - x^4$ by $a - x$.

5*

62. It may sometimes happen, that one or both of the quantities, proposed for division, may contain several terms affected with the same power of the letter, in reference to which the arrangement is made. The following examples will exhibit the course to be pursued in cases of this kind.

1. To divide

$11a^2b - 19abc + 10a^3 - 15a^2c + 3ab^2 + 15bc^2 - 5b^2c$

by $5a^2 + 3ab - 5bc$.

The terms $11a^2b - 15a^2c$ may be put under the form $(11b - 15c)a^2$, or which is the more convenient method

$$\left.\begin{array}{r} 11b \\ -15c \end{array}\right| a^2,$$

a vertical line being employed instead of a parenthesis to indicate that the quantities $11b, -15c$, placed one under the other at the left hand, are multiplied each by a^2. In like manner, the terms $-19abc + 3ab^2$ may be put under the form $\left.\begin{array}{r} -19bc \\ 3b^2 \end{array}\right| a$

Arranging the quantities with reference to the letter a, the calculations may be performed as follows.

$$\begin{array}{rl|rl|l|l}
10a^3 + 11b & a^2 & - 19bc & a & - 5b^2c + 15bc^2 & 5a^2 + 3ab - 5bc \\
-15c & & +3b^2 & & & \overline{2a + b - 3c} \\
10a^3 + 6b & a^2 & - 10bc & a & & \\
\hline
\text{1st Rem.} & & & & & \\
+5b & a^2 & -9bc & a & - 5b^2c + 15bc^2 & \\
-15c & & 3b^2 & & & \\
+5b & a^2 & -9bc & a & - 5b^2c + 15bc^2 & \\
-15c & & 3b^2 & & & \\
\hline
\text{2d Rem.} & & 0 & & &
\end{array}$$

Dividing first $10a^3$ by $5a^2$, we have $2a$ for the quotient; subtracting the product of the divisor by $2a$ from the dividend, we obtain the first remainder; dividing the part affected with a^2 in this remainder by $5a^2$, we obtain $b - 3c$ for the quotient; multiplying successively each term in the divisor by $b - 3c$, we exhaust the dividend; whence the quotient is $2a + b - 3c$.

2.
$$\begin{array}{r|r|r|l} 12b^2 & a^3+23b^3 & a^2+10b^4 & a \\ -29bc & -31b^2c & -6b^2c^2 & \\ 15c^2 & -9bc^2 & & \\ & 15c^3 & & \end{array}\left\}\begin{array}{r|l} 3b & a+2b^2 \\ -5c & \\ \hline 4b & a^2+5b^2 \;\Big|\; a \\ -3c & \quad -3c^2 \end{array}\right.$$

$$\begin{array}{lr|r|l} \text{1st Rem.} & 15b^3 & a^2+10b^4 & a \\ & -25b^2c & -6b^2c^2 & \\ & -9bc^2 & & \\ & 15c^3 & & \\ \hline \text{2d Rem.} & 0 & & \end{array}$$

The following are the partial divisions required in this question.

First partial division.

$$\begin{array}{r} 12b^2-29bc+15c^2 \\ \hline -29bc+15c^2 \\ \hline 0 \end{array}\left\}\begin{array}{l} 3b-5c \\ \hline 4b-3c \end{array}\right.$$

Second partial division.

$$\begin{array}{r} 15b^3-25b^2c-9bc^2+15c^3 \\ \hline -9bc^2+15c^3 \\ \hline 0 \end{array}\left\}\begin{array}{l} 3b-5c \\ \hline 5b^2-3c^2 \end{array}\right.$$

3. In like manner the following example may be performed.

1. To divide

$$\begin{array}{r|r|r|l} -a^6-b^2 & a^4+b^4 & a^2+b^6 & \text{by } a^2-b^2-c^2 \\ 2c^2 & -c^4 & 2b^4c^2 & \\ & & b^2c^4 & \end{array}$$

$$\text{Answer } \begin{array}{r|l} -a^4-2b^2 & a^2-b^4 \\ c^2 & -b^2c^2 \end{array}$$

63. If the dividend contain one or more letters, which are not found in the divisor, the quotient may be obtained in a more simple manner than by the common rule. Let it be required, for example, to divide

$3a^2b^3-3abc^3-2b^3c^2+b^5-3a^2bc^2+3ab^3c-a^2c^3+bc^4+a^2b^2c$ by b^2-c^2.

The dividend, being arranged in relation to a, may be put under the form

$(3b^3+b^2c-3bc^2-c^3)a^2+(3b^3c-3bc^3)a$
$+(b^5-2b^3c^2+bc^4)a^0$

Here in order that the division may be exact, it is evident, that the coefficients of the different powers of a must each be divisible by the divisor $a^2 - b^2$; the partial divisions required may therefore be indicated, thus,

$$\frac{3b^3 + b^2c - 3bc^2 - c^3}{b^2 - c^2},\quad \frac{3b^3c - 3bc^3}{b^2 - c^2},\quad \frac{b^5 - 2b^3c^2 + bc^4}{b^2 - c^2}$$

or, performing the operations, the partial quotients will be $3b + c$, $3bc$, and $b^3 - bc^2$; whence the quotient required will be $(3b + c)a^2 + 3bca + b^3 - bc^2$.

In general, *to divide a polynomial by a quantity, which does not contain the letter, with reference to which the polynomial is arranged, it is sufficient to perform the division upon each of the coefficients of this letter.*

64. When the dividend is not divisible by the divisor, we may still attempt the division, according to the rules which have been given, and continue it at pleasure.

Thus let it be required to divide x by $x + z$.

$$\begin{array}{l|l}
x & x + z \\
x + z & 1 - \frac{z}{x} + \frac{z^2}{x^2} - \frac{z^3}{x^3} + \frac{z^4}{x^4} \text{ \&c.} \\
\hline
-z & \\
-z - \frac{z^2}{x} & \\
\hline
\qquad \frac{z^2}{x} & \\
\qquad \frac{z^2}{x} + \frac{z^3}{x^2} & \\
\hline
\qquad\quad -\frac{z^3}{x^2} & \\
\qquad\quad -\frac{z^3}{x^2} - \frac{z^4}{x^3} & \\
\hline
\qquad\qquad \frac{z^4}{x^3} & \\
\qquad\qquad \frac{z^4}{x^3} + \frac{z^5}{x^4} & \\
\hline
\qquad\qquad -\frac{z^5}{x^4} \text{ \&c.} &
\end{array}$$

From the number of terms in the quotient already obtained in the above example, the learner will readily infer a law, by which the quotient may be continued at pleasure without performing any more operations.

65. If $a^2 - b^2$ be divided by $a - b$, we obtain an exact quotient $a + b$. So also $a^3 - b^3$ divided by $a - b$ gives an exact quotient $a^2 + a b + b^2$. In like manner $a^4 - b^4$ divided by $a - b$ gives an exact quotient. We therefore readily infer by analogy, that whatever may be the exponents of a and b, the division may be exactly performed.

In order to show in a rigorous manner, that this will be the case, let m represent any entire number whatever, and putting $m + 1$ for the exponent of the letters a and b, let us attempt the division of $a^{m+1} - b^{m+1}$ by $a - b$.

$$\begin{array}{l|l} a^{m+1} - b^{m+1} & a - b \\ \underline{a^{m+1} - a^m b} & \overline{a^m} \end{array}$$

first remainder $\quad a^m b - b^{m+1}$

or $\quad b(a^m - b^m)$

Dividing a^{m+1} by a, we have by the rule for the exponents a^m for the first term of the quotient. Multiplying next the divisor $a - b$ by a^m, and subtracting the product from the dividend, we have for the first remainder

$$a^m b - b^{m+1}$$

an expression, which may be put under the form $b(a^m - b^m)$.

Without proceeding further, it is evident, that if $a^m - b^m$ is exactly divisible by $a - b$, $a^{m+1} - b^{m+1}$ will also be exactly divisible by $a - b$, that is, if the difference of two similar powers of any degree whatever of two quantities is exactly divisible by the difference of these quantities, then, the difference of two powers of these same quantities, one degree more elevated, will also be divisible by the difference of these quantities.

But we have seen that $a^2 - b^2$ is divisible by $a - b$, it follows therefore from what has just been demonstrated, that $a^3 - b^3$ will be divisible by $a - b$, and if $a^3 - b^3$ be divisible by $a - b$, it follows then that $a^4 - b^4$ must be divisible by

$a - b$. Passing thus from one exponent of the letters a and b to another, we see that whatever may be the exponent of these letters, the division may be exactly performed.

66. Miscellaneous examples in the division of algebraic quantities.

1. To divide $10x^2y - 15y^2 - 5y$ by $5y$.
2. To divide $3a^2 - 15 + 6a + 3b$ by $3a$.
3. To divide $32x^6 - 8c^2x^3 + 12x^3 - c^2 + 1$ by $4x^3 - c^2 + 1$.
4. To divide $x^5 + 1$ by $x + 1$.
5. To divide $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$ by $1 - 2x + x^2$.
6. To divide $a^5 - 5a^4z + 10a^3z^2 - 10a^2z^3 + 5az^4 - z^5$ by $a^2 - 2az + z^2$.
7. To divide $6x^6 - 5x^5y^2 + 21x^3y^3 - 6x^4y^4 + x^2y^5 + 15y^6$ by $2x^3 - 3x^2y^2 + 5y^3$.
8. To divide 1 by $1 - a$.
9. To divide $b^2a^3 - c^2a^3 + b^4a^2 - c^4a^2 + 3ba + 3ca + b^2 - c^2$ by $b + c$.
10. To divide $\begin{array}{r|r|r|r|r} 6b & a^4 - 7b^2 & a^3 - 3b^3 & a^2 + 4b^3 & a + b^2 \\ -10 & 23b & 22b^2 & -9b^2 & -2b \\ & -20 & -31b & 5b & \\ & & 5 & -5 & \end{array}$

by $\begin{array}{r|l} 3b & a + b^2 - 2b \\ -5 & \end{array}$

ALGEBRAIC FRACTIONS.

67. When the division of two algebraic quantities cannot be exactly performed, the quotient, as we have seen, is expressed in the form of a fraction, the dividend being taken for the numerator and the divisor for the denominator.

A fraction in algebra has the same signification as a fraction in arithmetic; the denominator shows into how many parts unity is divided and the numerator how many of these parts are taken. Thus, in the algebraic fraction $\frac{a}{b}$, uni-

ty is supposed to be divided into b parts, and a number a of these parts is supposed to be taken.

REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS.

68. A fraction is said to be in its lowest terms, when there is no quantity, that will divide both of its terms without a remainder. To reduce a fraction therefore to this state, we suppress in the numerator and denominator the factors, which are common to them.

When the two terms of an algebraic fraction are simple quantities, it will be easy from inspection to determine the factors common to them; but if the terms of the fraction are polynomials, this will not be so easy, and we must in this case have recourse to the method of the *greatest common divisor.*

By the greatest common divisor of two algebraic quantities we understand *the greatest in regard to coefficients and exponents, that will exactly divide these quantities.* Its theory rests upon the same two principles, as that of the greatest common divisor in arithmetic, viz.

1°. *The greatest divisor common to two quantities contains as factors all the particular divisors common to these quantities and no others.* 2°. *The greatest divisor common to two quantities must divide the remainder after their division.*

From the first of these principles, it follows, *That we may multiply or divide either of the quantities proposed by any quantity, which is not itself a factor or which does not contain a factor common to the other, without affecting the greatest common divisor of these quantities.*

The greatest common divisor, for example, of the two quantities $a\,b$, $a\,c$ is a; If we now multiply one of them $a\,b$, for instance, by d it becomes $a\,b\,d$; but the greatest common divisor of $a\,b\,d$ and $a\,c$ is evidently the same as that of $a\,b$ and $a\,c$.

69. This being premised, let it be proposed to find the greatest common divisor of the polynomials

$$a^3 - a^2 b + 3 a b^2 - 3 b^3 \text{ and } a^2 - 5 a b + 4 b^2$$

Pursuing the same general course as in arithmetic, we commence by dividing the first of the proposed polynomials by the second; we thus obtain $a + 4 b$ for a quotient with a remainder $19 a b^2 - 19 b^3$.

By the second of the above principles the question is now reduced to finding the greatest common divisor to this remainder and the divisor $a^2 - 5 a b + 4 b^2$. But $19 a b^2 - 19 b^3$ may be put under the form $19 b^2 (a - b)$; and since the factor $19 b^2$ of this quantity is not a factor of $a^2 - 5 a b + 4 b^2$, we may by virtue of the first of the above principles reject it; the question is therefore still further reduced to finding the greatest common divisor to $a - b$ and $a^2 - 5 a b + 4 b^2$.

Dividing the last of these two quantities by the first we obtain an exact quotient $a - 4 b$; whence $a - b$ is their greatest common divisor; and by consequence it is the greatest common divisor of the polynomials proposed.

The following is a table of the calculations.

$$\begin{array}{lll}
\text{1st operation} & a^3 - a^2 b + 3 a b^2 - 3 b^3 & \left\{ \begin{array}{l} a^2 - 5 a b + 4 b^2 \\ \hline a + 4 b \end{array} \right. \\
 & a^3 - 5 a^2 b + 4 a b^2 & \\
\hline
 & \quad 4 a^2 b - a b^2 - 3 b^3 & \\
 & \quad 4 a^2 b - 20 a b^2 + 16 b^3 & \\
\hline
 & \qquad\quad 19 a b^2 - 19 b^3 & \\
\text{or} & \qquad\quad 19 b^2 (a - b) & \\
\end{array}$$

$$\begin{array}{ll|l}
\text{2d operation} & a^2 - 5 a b + 4 b^2 & a - b \\
\cline{3-3}
 & a^2 - a b & a - 4 b \\
\cline{2-2}
 & \quad - 4 a b + 4 b^2 & \\
 & \quad - 4 a b + 4 b^2 & \\
\cline{2-2}
 & \qquad 0 & \\
\end{array}$$

70. Let us now resume the same example, arranging the quantities with reference to the letter b. They will then

stand, thus, $-3b^3 + 3ab^2 - a^2b + a^3$, $4b^2 - 5ab + a^2$. Here the first term $-3b^3$ of the dividend is not divisible by $4b^2$, the first term of the divisor. It will be observed, however, that 4 the coefficient of the first term of the divisor is not a factor of the divisor. We may therefore, in virtue of the first principle, multiply the dividend by 4, without affecting the greatest common divisor required. Performing this operation, we have for the dividend

$$-12b^3 + 12ab^2 - 4a^2b + 4a^3.$$

Dividing next $-12b^3$ by $4b^2$, we obtain $-3b$ for a quotient. Multiplying the whole divisor by $-3b$, and subtracting, we have for a remainder $-3ab^2 - a^2b + 4a^3$.

The exponent of b in this remainder, being equal to the exponent of the same letter in the divisor, we continue the operation; and in order to render the first term divisible by the first term of the divisor, we multiply anew by 4, which gives $-12ab^2 - 4a^2b + 16a^3$. Dividing this by the divisor, the quotient is $-3a$, which we separate from the other by a comma, to show that it has no connexion with it, and the remainder is $-19a^2b + 19a^3$, or $19a^2(-b + a)$.

Suppressing, as before, the factor $19a^2$, the question is reduced to finding the greatest divisor common to $-b + a$ and $4b^2 - 5ab + a^2$. Dividing therefore the last of these quantities by the first, we obtain an exact quotient $-4b + a$; whence $-b + a$, or which is the same thing $a - b$ is the greatest common divisor sought.

See a table of the calculations.

1st operation

$$\begin{array}{rl}
-12b^3 + 12ab^2 - 4a^2b + 4a^3 & \Big\} \; \underline{4b^2 - 5ab + a^2} \\
\underline{-12b^3 + 15ab^2 - 3a^2b \qquad\quad} & \Big\} \; -3b, \qquad -3a \\
-3ab^2 - a^2b + 4a^3 & \\
-12ab^2 - 4a^2b + 16a^3 & \\
\underline{-12ab^2 + 15a^2b - 3a^3} & \\
-19a^2b + 19a^3 & \\
\text{or} \quad 19a^2(-b + a) &
\end{array}$$

$$\text{2d operation}\quad \begin{array}{l} 4b^2-5ab+a^2 \\ 4b^2-4ab \\ \hline -ab+a^2 \\ -ab+a^2 \\ \hline 0 \end{array} \left\{ \begin{array}{l} -b+a \\ \hline -4b+a \end{array} \right.$$

In this example, as in the case of all others, in which the exponent of the principal letter in the dividend exceeds by unity the exponent of the same letter in the divisor, by multiplying the dividend at the outset by the square of the coefficient of the first term of the divisor, the division may be continued, without further preparation, until a remainder is obtained, of a less degree in regard to the principal letter, than the divisor.

If the exponent of the principal letter in the dividend exceeds by two, three, &c. units the exponent of the same letter in the divisor, to avoid further preparation, the dividend must be multiplied by the third fourth &c. powers of the coefficient of the first term of the divisor.

71. The suppression of the factor $19b^2$, in the first remainder of the preceding example, serves not only to simplify the calculations, but it is also indispensable ; for unless this be done, we must multiply the new dividend by $19b^2$, in order to render the first term divisible by the first term of the divisor ; we shall thus introduce into this dividend a factor, which is found in the divisor, and by consequence we shall introduce into the greatest common divisor sought a factor, which does not belong to it.

72. Let it be proposed next to find the greatest common divisor of the polynomials

$$15a^5+10a^4b+4a^3b^2+6a^2b^3-3ab^4$$

and $$12a^3b^2+38a^2b^3+16ab^4-10b^5.$$

Before proceeding to the division of the proposed polynomials, we observe that the first contains the letter a as a factor common to all its terms ; and since this letter does not enter as a factor into the second polynomial, we may

suppress it, as forming no part of the greatest common divisor sought.

For a similar reason, the factor $2b^2$ may be suppressed in the second polynomial. Thus the question is reduced to finding the greatest common divisor of the polynomials

$$15\,a^4 + 10\,a^3\,b + 4\,a^2\,b^2 + 6\,a\,b^3 - 3\,b^4$$

and $$12\,a^3 + 38\,a^2\,b + 16\,a\,b^2 - 10\,b^3$$

Pursuing with these polynomials the same course, as in the preceding example, we obtain for the greatest common divisor $3\,a^2 + 2\,a\,b - b^2$.

73. From what has been done, we have the following rule, by which to find the greatest common divisor of two polynomials, viz. The polynomials proposed being arranged with reference to the same letter, 1°. *We suppress in each the factors, which are not found in the other*; 2°. *we divide one of the polynomials by the other, and if the division can not be exactly performed, we divide the first divisor by the remainder, and so on, observing to prepare each dividend when necessary in such a manner, as to render the first term divisible by the first term of the divisor, and to suppress in each remainder the factors, which are not contained in the preceding divisor; and that remainder, which will exactly divide the preceding, will be the greatest common divisor sought.*

EXAMPLES.

1. To find the greatest common divisor of the polynomials

$$5\,a^3 - 18\,a^2\,b + 11\,a\,b^2 - 6\,b^3$$
$$7\,a^2 - 23\,a\,b + 6\,b^2$$

2. To find the greatest common divisor of the polynomials

$$6\,x^5 - 4\,x^4 - 11\,x^3 - 3\,x^2 - 3\,x - 1$$
$$4\,x^4 + 2\,x^3 - 18\,x^2 + 3\,x - 5$$

74. The research of the greatest common divisor of two polynomials admits, in certain cases, of simplifications, which we shall now explain.

1. Let it be proposed to find the greatest common divisor of the polynomials

$$5\,a^6 + 10\,a^5\,x + 5\,a^4\,x^2$$

and $$a^4\,x + 2\,a^3\,x^2 + 2\,a^2\,x^3 + a\,x^4$$

* $(2b^2)$

† [illegible]

The letter a, it will be perceived, enters as a factor into each of the terms of the polynomials proposed. This letter will therefore be a factor of the greatest common divisor sought. Suppressing a in the proposed, and applying the rule to the polynomials, which result, we obtain $a + x$ for their greatest common divisor. The greatest common divisor sought will therefore be $a(a + x)$ or $a^2 + ax$.

2. Let it be required next to find the greatest common divisor of the polynomials

$$\begin{array}{r|l|l|l} b^2 & a^6 + b^3 & a^5 + b^4c^2 & a^2 \\ -c^2 & \quad\;\; -bc^2 & \quad\;\;\; -b^2c^4 & \end{array}$$

$$\begin{array}{r|l|l|l} b & a^4 + b^2 & a^3 + b^3 & a^2 \\ -c & \quad\; -bc & \quad\;\; -b^2c & \end{array}$$

The proposed, it will readily be perceived, have a simple factor a^2 common to both; recollecting that this will form a part of the greatest common divisor sought, we suppress it in the proposed, and the polynomials, which result will be

$$\begin{array}{r|l|l} b^2 & a^4 + b^3 & a^3 + b^4c^2 \\ -c^2 & \quad\;\; -bc^2 & \quad\;\;\; -b^2c^4 \end{array}$$

$$\begin{array}{r|l|l} b & a^2 + b^2 & a + b^3 \\ -c & \quad\; -bc & \quad\; -b^2c \end{array}$$

We may now commence the division of one of these polynomials by the other according to the rule, in order to determine their greatest common divisor. Before proceeding to this, however, let us see if there be not a polynomial divisor common to the coefficients of the letter a, with reference to which the arrangement is made.

Comparing for this purpose the two coefficients of the lowest degree $b^2 - c^2$ and $b - c$, we find that $b - c$ will divide both without a remainder. We inquire next if $b - c$ will divide the remaining coefficients of a. This is the case; $b - c$ therefore is a divisor common to all the coefficients of the two last polynomials. Recollecting that $b - c$ also will form a part of the greatest common divisor sought, we suppress $b - c$, and the polynomials which result will be

$$\begin{array}{r|l|l} b & a^4 + b^2 & a^3 + b^3c^2 + b^2c^3, \text{ and } a^2 + ba + b^2 \\ c & \quad\;\; bc & \end{array}$$

Applying the rule to these, the first, it will be perceived, contains a factor $b+c$, which is not contained in the second. Suppressing this, it remains to find the greatest common divisor of the polynomials

$$a^4 + b\,a^3 + b^2\,c^2$$
$$a^2 + b\,a + b^2$$

These, it will soon be found, have no common divisor. The greatest common divisor of the proposed will therefore be $a^2\,(b-c)$ or $a^2\,b - a^2\,c$.

3. Let it be proposed, as a third example, to find the greatest common divisor of the polynomials

$$\begin{array}{r|l|l|l} y^3 & x^4 + b\,y^3 & x^3 + b\,c\,y^3 & x^2 \\ -y\,z^2 & \quad c\,y^3 & \quad -b\,c\,y\,z^2 & \\ & -b\,y\,z^2 & & \\ & -c\,y\,z^2 & & \end{array}$$

$$\begin{array}{r|l|l} y^2\,z & x^2 + b\,y^2\,z & x + b\,d\,y^2\,z \\ -y\,z^2 & \quad d\,y^2\,z & \quad -b\,d\,y\,z^2 \\ & -b\,y\,z^2 & \\ & -d\,y\,z^2. & \end{array}$$

The simple quantity $x\,y$, it will be perceived, will exactly divide each of the terms of the first of the proposed polynomials, and $y\,z$ those of the second. Setting apart therefore the factor y, common to the quantities $x\,y$ and $y\,z$, as forming a part of the greatest common divisor sought, and dividing the first of the proposed by $x\,y$ and the second by $y\,z$, the polynomials, which result will be

$$\begin{array}{r|l|l|l} y^2 & x^3 + b\,y^2 & x^2 + b\,c\,y^2 & x \\ -z^2 & \quad c\,y^2 & \quad -b\,c\,z^2 & \\ & -b\,z^2 & & \\ & -c\,z^2 & & \end{array}$$

$$\begin{array}{r|l|l} y & x^2 + b\,y & x + b\,d\,y \\ -z & \quad d\,y & \quad -b\,d\,z \\ & -b\,z & \\ & -d\,z. & \end{array}$$

The coefficients of the first of these are divisible each by $y^2 - z^2$, and those of the second by $y - z$; since then $y - z$ is a divisor of $y^2 - z^2$, we set apart $y - z$ also as a

part of the greatest common divisor sought; and suppressing in the first the factor $y^2 - z^2$, and in the second $y - z$, the polynomials, which result will be

$$x^3 + \left.\begin{matrix} b \\ c \end{matrix}\right| x^2 + b\,c\,x, \text{ and } x^2 + \left.\begin{matrix} b \\ d \end{matrix}\right| x + b\,d$$

Applying the rule to these last, we obtain $x + b$ for their greatest common divisor; The greatest common divisor of the proposed will therefore be $y\,(y - z)\,(x + b)$.

75. From what has been done, the following method for finding the greatest common divisor of two polynomials will be readily inferred, viz.

1°. Suppress in the polynomials proposed the greatest simple divisors, which they respectively contain, observing to set aside as a part of the greatest common divisor sought, the factors which these divisors have in common. 2°. Suppress in the polynomials, which result, the greatest polynomial divisors independent of the principal letter, which they respectively contain, and set aside, as a part of the greatest common divisor sought, the factors, which these divisors have in common. 3°. Find the greatest common divisor of the polynomials, which result; and the product of this into each of the parts already obtained will be the greatest common divisor sought.

MISCELLANEOUS EXAMPLES.

1. To find the greatest common divisor of the polynomials

$$6\,a^5 + 5\,a^4\,b - 6\,a^3\,c^2 - 5\,a^2\,b\,c^2$$
$$a^3\,b - a^2\,b\,c - a\,b\,c^2 + b\,c^3$$

2. To find the greatest common divisor of the polynomials

$$x^6 + 4\,x^5 - 3\,x^4 - 16\,x^3 + 11\,x^2 + 12\,x - 9$$
$$6\,x^5 + 20\,x^4 - 12\,x^3 - 48\,x^2 + 22\,x + 12$$

3. To find the greatest common divisor of the polynomials

$$36\,a^6 - 18\,a^5 - 27\,a^4 + 9\,a^3$$
$$27\,a^5\,b^2 - 18\,a^4\,b^2 - 9\,a^3\,b^2$$

4. To find the greatest common diviscr of the polynomials

$$\begin{array}{r|r}
x^4 - 3x^3 - 5y^2 & x^2 + 12y^2 x + 4y^4 \\
y & -4y^3 \\
2 & -8y^2
\end{array}$$

$$\begin{array}{r|r|r}
x^3 - 3y & x^2 - y^2 & x + 3y^3 \\
-3 & 10y & -3y^2 \\
 & 2 & -6y
\end{array}$$

5. To find the greatest common divisor of the polynomials

$$\begin{array}{r|r|r|r|r|l}
y^3 & x^7 - 3y^3 & x^6 - 5y^5 & x^5 + 12y^6 & x^4 + 4y^7 & x^3 \\
y^2 & -3y^2 & -4y^4 & 12y^4 & -12y^5 & \\
 & & 3y^3 & & -8y^4 & \\
 & & 2y^2 & & &
\end{array}$$

$$\begin{array}{r|r|r|r|l}
y^3 & x^5 - 3y^4 & x^4 - y^6 & x^3 + 3y^6 & x^2 \\
-y & -3y^3 & 10y^4 & -3y^5 & \\
 & 3y^2 & 3y^3 & -9y^4 & \\
 & 3y & -10y^2 & 3y^3 & \\
 & & -2y & 6y^2 &
\end{array}$$

76. To reduce a fraction to its lowest terms, we divide the two terms of the fraction by their greatest common divisor.

EXAMPLES.

1. Reduce $\dfrac{x^4 - b^4}{x^5 - b^2 x^3}$ to its lowest terms.

2. Reduce $\dfrac{x^4 - 5x^2 + 4}{x^5 - 3x^4 - 11x^3 + 27x^2 + 10x - 24}$ to its lowest terms.

3. Reduce $\dfrac{3a^3 - 24a - 9}{2a^3 - 16a - 6}$ to its lowest terms.

4. Reduce $\dfrac{36a^6 - 18a^5 - 27a^4 + 9a^3}{27a^5 b^2 - 18a^4 b^2 - 9a^3 b^2}$ to its lowest terms.

77. Algebraic fractions being of the same nature as fractions in arithmetic, the rules for the fundamental operations are the same. We shall merely subjoin these rules, with some examples under each.

MULTIPLICATION OF ALGEBRAIC FRACTIONS.

Rule. *Multiply the numerators together for a new numerator, and the denominators for a new denominator.*

EXAMPLES.

1. Multiply $\frac{5c}{a^2}$ by $\frac{a^2b^2}{5cx^2}$.

2. Multiply $\frac{a^2+2ab+b^2}{cd-d^2}$ by $\frac{d^2}{a+b}$.

3. Multiply $\frac{4x-12}{5x}$ by $\frac{x+3}{4}$.

4. Multiply $\frac{a^2+b^2}{c+d}$ by $\frac{a-b}{c-d}$

DIVISION OF ALGEBRAIC FRACTIONS.

Rule. *Invert the divisor, and then proceed as in multiplication.*

EXAMPLES.

1. Divide $\frac{a+b}{x-y}$ by $\frac{x-y}{a-b}$.

2. Divide $\frac{3ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$.

3. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$

ADDITION OF ALGEBRAIC FRACTIONS.

Rule. *Reduce the fractions to a common denominator; then add the numerators together, and place their sum over the common denominator.*

EXAMPLES.

1. Add $\frac{3x}{5a}$, $\frac{x}{4b}$ and $\frac{2x}{c}$ together.

2. Add $\frac{3x^2+5y^3}{4y}$ and $\frac{5x^3+3y^2}{2y}$ together.

3. Add $\frac{x^2+y^2}{a-b}$ and $\frac{a+b}{x-y}$ together.

4. Add $\frac{2x+3}{5}$, $\frac{3x-1}{2x}$, and $\frac{4x}{7}$ together.

SUBTRACTION OF ALGEBRAIC FRACTIONS.

Rule. *Reduce the fractions to a common denominator; then place the difference of their numerators over the denominator, and it will be the difference required.*

EXAMPLES.

1. From $\frac{c+d}{a+b}$ take $\frac{a-2d}{a-b}$.

2. From $\frac{3x+1}{x+1}$ take $\frac{4x}{5}$.

3. From $\frac{2y^2-2y+1}{y^2-y}$ take $y-1$.

4. From $\frac{3x+y}{5a+x}$ take $\frac{a-2y}{a-x}$.

SECTION IV. EQUATIONS.

78. The above rules are sufficient for the solution of all equations of the first degree, however complicated.

Let it be proposed, for example, to find the value of x in the equation.

$$\frac{10x+bc}{a} - \frac{12x-c^2}{13a-b} = \frac{9x+3bc}{b}.$$

Indicating the operations required in order to make the denominators disappear, we have,

$$(10x+bc)(13a-b)b-(12x-c^2)ab = a(9x+3bc)(13a-b)$$

performing the operations, transposing and reducing, we have

$$127abx-117a^2x-10b^2x = 39a^2bc-16ab^2c+b^3c+abc^2$$

whence $x = \frac{39a^2bc-16ab^2c+b^3c-abc^2}{127ab-117a^2-10b^2}$

The following examples will serve as an exercise for the learner.

1. Given $\frac{7x-8}{11}+\frac{15x+8}{13}=3x-\frac{31-x}{2}$, to find the value of x.

2. Given $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$, to find the value of x. $= \frac{140607}{1962}$

3. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$, to find the value of x.

4. Given $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$, to find the value of x.

5. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$, to find the value of x.

PROBLEMS AND EQUATIONS OF THE FIRST DEGREE WITH TWO OR MORE UNKNOWN QUANTITIES.

79. Most of the questions, which we have hitherto considered, involve more than one unknown quantity. We have, however, been able to solve these questions by representing one of the unknown quantities only by a letter, since, by means of this it has been easy from the conditions of the question to express the other unknown quantity. In many questions the solution becomes more simple by representing more than one of the unknown quantities by a letter, and in complicated questions, it is frequently necessary to do this.

The question art. 1 viz. *To divide the number* 18 *into two such parts, that the greater may exceed the less by* 4, presents itself naturally with two unknown quantities. Thus, denoting the lesser part by x and the greater by y, we have by the conditions of the question.

$$x+y=18$$
$$y-x=4$$

Deducing the value of y from the second equation, we

have $y = x + 4$; substituting for y in the first equation its value $x + 4$, we have $x + x + 4 = 18$, an equation, which contains only one unknown quantity, and from which we obtain $x = 7$.

A person has two horses and a saddle, which of itself is worth $10. If the saddle be put upon the first horse, his value will be twice the second; but if the saddle be put upon the second, his value will be three times the first. What is the value of each?

Let $x =$ the value of the first horse, and y that of the second, we have by the question

$$x + 10 = 2y$$
$$y + 10 = 3x$$

Deducing the value of y from the second of these equations, and substituting it for y in the first, we have

$$x + 10 = 6x - 20$$

whence $$x = 6$$

Substituting next its value 6 for x in the second equation, we have $$y + 10 = 18$$
whence $$y = 8$$

The process, by which one of the unknown quantities in an equation is made to disappear, is called *elimination*. The method of eliminating one of the unknown quantities, pursued above, is called elimination by *substitution*.

80. Since the two members of an equation are equal quantities, it is evident, 1°. *that we may add two equations member to member without destroying the equality.* 2°. *we may subtract the members of one equation from those of another without destroying the equality.*

Taking advantage of this remark, we may frequently eliminate one of the unknown quantities in a more simple manner, than by the process of substitution.

Let there be proposed, for example, the equations

$$5x + 7y = 43$$
$$11x + 9y = 69$$

If either of the unknown quantities in these equations

were affected with the same coefficient we might, it is evident, eliminate this unknown quantity by a simple subtraction. But if the first equation be multiplied by 9 the coefficient of y in the second, and the second by 7 the coefficient of y in the first, we shall obtain two new equations, which may be substituted for the proposed, and in which the coefficients of y will be equal, viz.

$$45\,x + 63\,y = 387$$
$$77\,x + 63\,y = 483$$

Subtracting then the first of these equations from the second, we have $32\,x = 96$, from which we obtain $x = 3$.

In like manner, if we multiply the first of the proposed equations by 11 the coefficient of x in the second, and the second by 5 the coefficient of x in the first, we shall obtain two new equations, which may be substituted for the proposed, and in which the coefficients of x will be equal, viz.

$$55\,x + 77\,y = 473$$
$$55\,x + 45\,y = 345$$

Subtracting therefore the second of these equations from the first, we have $32\,y = 128$; whence $x = 4$.

Let us take as a second example the equations

$$8\,x - 21\,y = 33$$
$$6\,x + 35\,y = 177$$

The coefficients of x in these equations have, it will be perceived, a common factor 2. It will be sufficient therefore, in order to render these coefficients equal, to multiply the first equation by 3 and the second by 4. Performing the operations we have

$$24\,x - 63\,y = 99$$
$$24\,x + 140\,y = 708.$$

whence, subtracting the first of these equations from the second, we obtain

$$203\,y = 609$$

therefore $\qquad y = 3$

In like manner, since the coefficients of y contain the common factor 7, in order to render the coefficients of y equal

we multiply the first of the proposed equations by 5 and the second by 3, which gives the two new equations

$$40x - 105y = 165$$
$$18x + 105y = 531$$

whence by addition, we obtain

$$58x = 696$$

therefore $$x = 12$$

81. The method of elimination, which we have now explained, is called elimination *by addition and subtraction*, since, the equations being properly prepared, we cause one of the unknown quantities to disappear by addition or subtraction.

In the use of this method, it is important to ascertain whether the coefficients have common factors, since, if this be the case, by omitting the common factors in the multiplications required, the calculations to be performed become more simple.

EXAMPLES.

1. To find the values of x and y in the equations

$$\frac{3x-1}{5} + 3y - 4 = 15$$

$$\frac{3y-5}{6} + 2x - 8 = 7\tfrac{2}{3}$$

2. To find the values of x and y in the equations

$$\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5$$

$$\frac{5y-7}{2} + \frac{4x-3}{6} = 18 - 5x$$

3. To find the values of x and y in the equations

$$x + 1 - \frac{3y+4x}{7} = 7 - \frac{9y+33}{14}$$

$$y - 3 - \frac{5x-4y}{2} = x - \frac{11y-19}{4}$$

4. To find the values of x and y in the equations

$$\frac{7x-21}{6}+\frac{3y-x}{3}=4+\frac{3x-19}{2}$$

$$\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}$$

82. Let now the following question be proposed, viz.

There are three persons A, B, and C, whose ages are as follows; if from 4 times A's age added to 5 times B's age, we subtract 3 times C's age, the remainder will be 70; if from 3 times A's age we subtract 4 times B's age, and to the remainder add twice C's age, the sum will be 25; and if twice A's age, 3 times B's, and 5 times C's age be added together, the sum will be 240. What is the age of each?

This question presents itself naturally with three unknown quantities. Thus denoting A's age by x, B's by y, and C's age by z, we have by the question

$$\begin{aligned} 4x+5y-3z&=70\\ 3x-4y+2z&=25\\ 2x+3y+5z&=240 \end{aligned}$$

Multiplying the first equation by 2, and the second by 3, and adding the results, we obtain

$$17x-2y=215$$

Again, multiplying the second equation by 5, and the third by 2, and subtracting, we obtain

$$-11x+26y=355$$

We have now two equations with two unknown quantities only. Deducing next the values of x and y from these in the same manner as in the preceding equations with two unknown quantities, we have $x=15$, $y=20$; substituting these values in the first of the proposed equations, we obtain $z=30$.

In the same manner, if there be four equations with four unknown quantities, we combine the equations two by two, until one of the unknown quantities is eliminated from the whole; we then have three equations with three unknown

quantities. Combining next these three two by two, until one of the unknown quantities is eliminated, we obtain two equations with two unknown quantities, and so on. The process is altogether similar for five or more equations with the same number of unknown quantities.

EXAMPLES.

1. To find the values of x, y, and z in the equations

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46$$

2. To find the values of x, y, and z in the equations

$$3x + 5y + 7z = 179$$
$$8x + 3y - 2z = 64$$
$$5x - y + 3z = 75$$

3. To find the values of u, x, y, and z in the equations

$$x - 2y + 3z = 5u$$
$$3x - 15 - u = 4y - 23$$
$$2u + z - y = 27$$
$$y + 12 - 3x + 11u = 91$$

It sometimes happens, that all the unknown quantities are not found in each of the equations. In this case, the elimination may, with a little attention, be very readily performed.

Let there be proposed the four following equations with four unknown quantities, viz.

$$2x - 3y + 2z = 13$$
$$4u - 2x = 30$$
$$4y + 2z = 14$$
$$5y + 3u = 32$$

With a little examination we see, that the elimination of z from the first and third equations will give an equation in x and y, and that the elimination of u from the second and fourth equations will also give an equation in x and y. From these last the values of x and y may be readily found. Performing the necessary operations we obtain $x = 3$, $y = 1$.

Substituting next for x its value in the second equation, we have $u = 9$, and substituting for y its value in the third, we have $z = 5$.

Let it now be proposed to find the values of the unknown quantities in the following equations, viz.

$$\begin{aligned} 7x - 2z + 3u &= 17 \\ 4y - 2z + t &= 11 \\ 5y - 3x - 2u &= 8 \\ 4y - 3u + 2t &= 9 \\ 3z + 8u &= 33 \end{aligned}$$

83. We pass next to the solution of some questions.

1. A number consisting of two figures when divided by 4, gives a certain quotient, and a remainder of 3; when divided by 9 gives another quotient, and a remainder of 8. The value of the figure on the left hand is equal to the quotient obtained, when the number was divided by 9, and the other figure is equal to $\frac{1}{17}$ of the quotient obtained, when the number was divided by 4. Required the number.

Let x and y = the figures in order; then $10x + y$ = the number, and we have by the question

$$\frac{10x + y}{9} = x + \frac{8}{9}$$

$$\frac{10x + y}{4} = \frac{3}{4} + 17y$$

Deducing the values of x and y from these equations, we obtain $x = 7$, $y = 1$. The number required is therefore 71.

2. A merchant has three ingots composed each of gold, silver, and copper in the following proportions, viz. in every 16 oz. of the first there are 7 oz. of gold, 8 of silver, and one of copper; in every 16 oz. of the second there are 5 oz. of gold, 7 of silver, and 4 of copper; and in every 16 oz. of the third there are 2 oz. of gold, 9 of silver, and 5 of copper. What parts must be taken from each of these three ingots, in order to compose a fourth, in which there shall be $4\frac{11}{16}$ oz. of gold, $7\frac{10}{16}$ of silver, and $3\frac{7}{16}$ of copper to every 16 oz.

Let x, y, and z represent the number of ounces, which

must be taken respectively from the three ingots to form the ingot required. Then, since to every 16 oz. of the first ingot there are 7 oz. of gold; to a number of ounces denoted by x there will be $\frac{7x}{16}$ oz. of gold. In like manner in y oz. of the second ingot there will be $\frac{5y}{16}$ oz. of gold; and in z oz. of the third there will be $\frac{3z}{16}$ oz. of gold. The proportion of gold in the ingot required will then be

$$\frac{7x}{16}+\frac{5y}{16}+\frac{2z}{16}$$

Finding, in like manner, the proportions of silver and copper, we shall have for the equations of the question

$$\frac{7x+5y+2z}{16}=4\tfrac{15}{16}$$

$$\frac{8x+7y+9z}{16}=7\tfrac{10}{16}$$

$$\frac{x+4y+5z}{16}=3\tfrac{7}{16}$$

Deducing the values of x, y, and z from these equations, we have $x=4$, $y=9$ and $z=3$.

3. A sum of $500 was to be let out at simple interest, in two separate parts, the smaller at 2 per cent. more than the other. The interest of the greater part was afterwards increased, and that of the smaller diminished by one per cent. By this, the interest of the whole was augmented by one-fourth of the former value. But if the interest of the greater sum had been so increased, without any diminution of the less, the interest of the whole would have been increased one third. What were the two parts, and the rate per cent. of each ?

Let $x=$ the less part, $y=$ the rate of interest of the less; then $500-x=$ the greater, and $y-2=$ the rate of interest of the greater, and $\frac{xy}{100}+\frac{(500-x)(y-2)}{100}$ or

$5y - 10 + \frac{x}{50} =$ the first interest, and

$\frac{x(y-1)}{100} + \frac{(500-x)(y-1)}{100}$ or $5y - 5 =$ the second interest; we have therefore $5y - 5 = \frac{5}{4}\left(5y - 10 + \frac{x}{50}\right)$

Again, the third interest

$$= \frac{xy}{100} + \frac{(500-x)(y-1)}{100} = \frac{500y - 500 + x}{100},$$

we have therefore $\frac{500y - 500 + x}{100} = \frac{4}{3}\left(5y - 10 + \frac{x}{50}\right)$

Deducing the values of x and y from the equations above, we have $x = 100$, $y = 4$; the less part required is therefore \$100, and the greater \$400, and the rates of interest are \$4 and \$2 respectively.

84. The solution of a question is sometimes facilitated, by the introduction of a new unknown, auxiliary to the principal unknown quantities.

The ages of three persons are as follows. The age of A, added to 5 times the sum of the ages of B and C, is equal to 285; the age of B, added to 4 times the sum of the ages of A and C is 217; and the age of C, added to 3 times the sum of the ages of A and B is 163. Required the age of each.

Let x, y, and z represent their ages respectively, and let us introduce the unknown s to express the sum of their ages.

By the question then, we have

$$\begin{array}{lll} x + 5(s - x) = 285 & \text{or} & 5s - 4x = 285 \\ y + 4(s - y) = 217 & & 4s - 3y = 217 \\ z + 3(s - z) = 163 & & 3s - 2z = 163 \end{array}$$

making the coefficients of x, y, and z equal, we have

$$\begin{array}{l} 15s - 12x = 835 \\ 16s - 12y = 868 \\ 18s - 12z = 978 \end{array}$$

adding these last member to member, we have

$$49s - 12s = 2701$$

The value of s is therefore 73, by means of which we readily obtain $x=20$, $y=25$, $z=28$.

85. In what precedes, we have supposed the problem proposed to furnish as many distinct and independent equations, as there are unknown quantities employed. When this is not the case, the problem is said to be *indeterminate*, that is to say, the problem will admit of an *infinity* of solutions.

Indeed, let it be supposed that a problem with two unknown quantities gives but one equation only, for example, the equation

$$5x-3y=12$$

Deducing the value of x from this equation, we have

$$x=\frac{12+3y}{5}$$

Making successively in this last $\quad y= 1,\ 2,\ 3,\ 4,\ \ldots$

the corresponding values of x will be $x= 3,\ \frac{18}{5},\ \frac{21}{5},\ \frac{24}{5}\ \ldots,$

We thus have an infinity of systems of values for x and y, each of which, substituted for x and y, will equally satisfy the equation.

In like manner, if the conditions of the question furnish but two equations with three unknown quantities, eliminating one of the unknown quantities, we arrive at an equation with two unknown quantities, which will be satisfied by an infinity of systems of values for these unknown quantities, and from which we infer an infinity of values for the other unknown quantity.

86. The following examples will serve as an exercise in the solution of questions, involving two or more unknown quantities.

1. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

2. A vintner sold at one time 20 dozen of port wine, and

30 of sherry, and for the whole received \$120; and at another time sold 30 dozen of port, and 25 of sherry at the same prices as before, and for the whole received \$140. What was the price of a dozen of each sort of wine?

3. A farmer with 28 bushels of barley at 2*s.* 4*d.* per bushel, would mix rye at 3 shillings per bushel, and wheat at 4 shillings per bushel, so that the whole mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* per bushel. How many bushels of rye, and how many of wheat must he mix with the barley?

4. A and B speculate with different sums; A gains \$150, B loses \$50, and now A's stock is to B's as 3 to 2. But if A had lost \$50 and B gained \$100, then A's stock would have been to B's as 5 to 9. What was the stock of each?

5. A rectangular bowling green having been measured, it was observed, that if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader, and 5 feet longer, it would contain 113 feet more. Required the length and breadth.

6. A merchant mixes wheat, which costs him 10 shillings a bushel, with barley which costs him 4 shillings a bushel, in such proportion as to gain $43\frac{3}{4}$ per cent. by selling the mixture at 11 shillings a bushel. Required the proportion.

7. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{33}$ of it. How many will it hold of each?

8. There is a number consisting of two figures, the second of which is greater than the first; and if the number be divided by the sum of its figures, the quotient is 4; but if the figures be inverted, and the number which results be divided by a number greater by 2 than the difference of the figures, the quotient becomes 14. Required the number.

9. A person owes a certain sum to two creditors. At one time he pays them \$53, giving to one four-elevenths of the sum due to him, and to the other \$3 more than one-sixth of his debt to him. At a second time he pays them \$42, giving to the first three-sevenths of what remains due to him,

and to the other one-third of what is due to him. What were the debts?

10. A vintner has two casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After the casks become half empty, he puts 10 gallons of water into each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold?

11. A person having laid out a rectangular bowling green, observed that if each side had been 4 yds. longer, the adjacent sides would have been in the ratio of 5 to 4; but if each had been 4 yds. shorter, the ratio would have been 4 to 3. What are the lengths of the sides?

12. A has a capital of \$30 000, which he puts out to interest at a certain rate per cent. and he owes \$20 000, on which he pays a certain rate per cent. interest. The interest, which he receives exceeds that, which he pays, by \$800. B has a capital of \$35 000, which he puts out to interest at the same rate per cent. that A pays on his debt, he also owes \$24 000, on which he pays interest at the same rate, that A receives for his capital. The interest, which he receives exceeds that, which he pays by \$310. What are the two rates of interest?

13. Three guineas were to be raised on two estates to be charged proportionably to their values. Of this sum, A's estate, which was 4 acres more than B's, but worse by 2 shillings an acre, paid £1.15*s*. But had A possessed 6 acres more, and B's land been worth 3 shillings an acre less, it would have paid £2. 5*s*. Required the values of the estates.

14. A has a certain capital which he puts out to interest at a certain rate per cent. B has a capital of \$10 000 more than A, which he puts out to interest at one per cent. more, and receives \$800 more interest than A. C has a capital of \$15 000 more than A, which he puts out at 2 per cent. more, and receives \$1500 more interest than A. What is the capital of each and the three rates of interest?

15. Three brothers purchased an estate for \$15000, and the first wanted in order to complete his part of the payment half of the property of the second; the second would have paid his share with the help of a third of what the first owned; and the third required to make the same payment, in addition to what he had, a fourth part of what the first possessed; what was the amount of each one's property?

16. Three players after a game count their money; one had lost, the other two had gained each as much as he had brought to the play; after the second game, one of the players, who had gained before, lost, and the two others gained each a sum equal to what he had at the beginning of this second game; at the third game the player, who had gained till now, lost with each of the others a sum equal to that, which each possessed at the beginning of this last game; they then separated, each having \$1,20; how much had they each, when they commenced playing?

17. Three men A, B, C, driving their sheep to market says A to B and C, if each of you will give me 5 of your sheep, I shall have just half as many as both of you will have left. Says B to A and C, if each of you will give me 5 of yours I shall have just as many as both of you will have left. Says C to A and B, if each of you will give me 5 of yours I shall have just twice as many as both of you will have left. How many had each?

18. It is required to divide the number 72 into four such parts, that if the first part be increased by 5, the second part diminished by 5, the third part multiplied by 5, and the fourth part divided by 5, the sum, difference, product and quotient shall all be equal.

NEGATIVE QUANTITIES. QUESTIONS PRODUCING NEGATIVE RESULTS.

87. The length of a certain field is eight rods and the breadth five rods, how much must be added to the length, that the field may contain 30 square rods?

Let $x =$ the quantity to be added, then by the question

$$40 + 5\,x = 30$$

and $\qquad 5\,x = 30 - 40$

or dividing by 5 $\qquad x = 6 - 8$

In this result 8, the quantity to be subtracted, is greater than that, from which it is required to be taken; the subtraction therefore cannot be performed. We may, however, decompose 8 into two parts 6 and 2, the successive subtraction of which will be equal to that of 8, and we shall have for $6 - 8$ the equivalent expression $6 - 6 - 2$, which is reduced to $0 - 2$ or more simply -2, the sign $-$ being retained before the 2 to show that it remains to be subtracted.

A simple quantity with the sign $-$ prefixed is called a *negative* quantity, thus, -2, $-3\,a$, $-5\,a\,b$ are negative quantities.

Simple quantities with the sign $+$ either prefixed or understood are called positive quantities. Thus 2, $3\,a$, $5\,a\,b$, are positive quantities.

Negative quantities, it will be perceived, differ in nothing from positive quantities except in their sign. They are derived from endeavoring to subtract a larger quantity from one that is smaller, and are to be regarded merely as positive quantities to be subtracted.

88. If it now be asked what is the sum of the simple quantities $+a$, $-b$, $+c$, the question, from what has been said, is reduced to this, what change will be produced in the quantity a, if the quantity b be subtracted from it and the quantity c be added to the remainder. Indicating the operations required to obtain the answer to the question thus proposed, the result will be

$$a - b + c$$

In order then to add simple quantities affected with the signs $+$ and $-$, it will be sufficient *to write them one after the other with the signs with which they are affected, when they stand alone*.

89. If we now add the quantities $+b$, $-b$, the result

$b - b$, it is evident, will be equal to zero. If then the expression $b - b$ be added to a, it will not affect the value of a; and $a + b - b$ will only be a different form of expression for the same quantity a. If it now be proposed to subtract $+ b$ from a, it will be sufficient, it is evident, to efface $+ b$ in the equivalent expression $a + b - b$, and the result will be $a - b$. Again, if it be required to subtract $- b$ from a, it will be sufficient to efface $- b$ in the same expression, and we shall have for the result $a + b$. Thus, to subtract a positive quantity is the same as to add an equal negative quantity, and to subtract a negative quantity is the same as to add an equal positive quantity. To subtract simple quantities therefore of whatever sign, *we change the signs, and then proceed as in addition.*

90. If we multiply $b - b$ by a the product must be $a b - a b$, because the multiplicand being equal to zero, the product must be zero. Since then the product of b by a is evidently $a b$, that of $- b$ by a must be $- a b$ in order that the second term may destroy the first. For a similar reason the product of a by $b - b$ will be $a b - a b$; *Whence if a negative quantity be multiplied by a positive, or a positive by a negative, the product will be negative.*

Again, if we multiply $- a$ by $b - b$, from what has been proved above, the product of $- a$ by b will be $- a b$, the product of $- b$ by $- a$ must therefore be $+ a b$, in order that the result may be zero, as it should be, when the multiplier is zero. Whence, *the product of a negative quantity by a negative quantity will be positive.*

91. We arrive at the same conclusions by the definition of multiplication given in arithmetic, according to which *to multiply one number by another, we form a number by means of the multiplicand in the same manner that the multiplier is formed by means of unity.* Thus to multiply 5 by $- 3$, we form a number by means of 5 in the same manner that $- 3$ is formed by means of unity; but $- 3$ is formed by the subtraction of three units, the product required will therefore be formed by the subtraction of three fives, by consequence it will be

-15. In like manner the product of -5 by -3 will be found by the subtraction of three *minus* five's, it will therefore be $+15$.

The rules for division follow necessarily from those for multiplication. We have therefore the same rules for the signs in the multiplication and division of insolated simple quantities, as are applied to these quantities, when they make a part of polynomials; and in general, *simple quantities, when they are insolated, are combined in the same manner with respect to their signs, as when they make a part of polynomials.*

92. From what has been said, it will be perceived, that the term *addition* does not in algebra, as in arithmetic, always imply augmentation. Thus, the sum of a and $-b$ is, strictly speaking, the difference between a and b; it will therefore be less than a. To distinguish this from an arithmetical sum, we call it an *algebraic sum.* Thus the polynomial $3ab - 5bc + cd - ef$, considered as formed by uniting the quantities $3ab$, $-5bc$, $+cd$ with their respective signs is called an *algebraic sum.* Its proper acceptation is the arithmetical difference between the sum of the units contained in the terms, which are additive, and the sum of those contained in the terms, which are subtractive.

In like manner the term *subtraction* in algebra does not always imply diminution. Thus $-b$ subtracted from a gives $a + b$, which is greater than a. This result may, however, be called an *algebraic difference*, since it may be put under the form $a - (-b)$.

93. Resuming now the question proposed art. 87, we have for the answer $x = -2$. In order to interpret this negative result, we return to the equation of the question $40 + 5x = 30$. Here, the addition intended in the enunciation of the question being arithmetical, it is evidently absurd to require that something should be added to 40 in order to make 30, since 40 is already greater than 30. The negative result indicates therefore, that the question is arithmetically impossible, or in other words, that it cannot be solved in the exact sense of the enunciation. If, however,

in the equation $40 + 5x = 30$, we substitute -2 for x, we have $40 - 10 = 30$, an equation, which is exact. In order then that the result may be positive, or which is the same thing, that the question may be arithmetically possible, the enunciation should be modified, thus,

The length of a certain field is eight rods, and its breadth five rods; how much must be *subtracted* from the length, that the field may contain 30 square rods?

Putting x for the quantity to be subtracted, we have by this new enunciation $40 - 5x = 30$, from which we obtain $x = 2$.

2. The length of a certain field is 11 rods and its breadth 7 rods; how much must be subtracted from the length, that the field may contain 98 square rods?

Let $x =$ the quantity to be subtracted; then by the question
$$77 - 7x = 98$$
whence
$$x = -3$$

To interpret this result, we return to the equation of the question. Here, as an arithmetical subtraction is intended in the enunciation, it is evidently absurd to require, that something should be subtracted from 77 to make 98, since 77 is already less than 98. The question therefore cannot be solved in the exact sense of the enunciation. If however instead of x in the equation of the question, we substitute -3, we have $77 + 21 = 98$, an equation, which is exact. In order then that the result may be positive, the question should be modified, thus,

The length of a certain field is 11 rods and the breadth 7 rods; how much must be *added* to the length in order that the field may contain 98 square rods?

Resolving the question according to this new enunciation, we obtain $x = 3$.

Let us take as a third example the following question.

A laborer wrought for a person 12 days and had his wife and son with him 7 days, and received 46 shillings. He afterwards wrought 8 days, having his wife and son with him 5 days, and received 30 shillings; how much did he earn per day himself, and how much did his wife and son earn?

Let $x =$ the daily wages of the man, and y that of his wife and son; we have by the question

$$12x + 7y = 46$$
$$8x + 5y = 30$$

Resolving these equations, we obtain $x = 5$, $y = -2$.

In order to interpret the negative result, we substitute 5 for x in the equations above, by which we have

$$60 + 7y = 46$$
$$40 + 5y = 30$$

equations, which are evidently absurd, since it is required to add something to 60 in order to make 46, and to 40 in order to make 30. If, however, we substitute -2 for y in these last, we have

$$60 - 14 = 46$$
$$40 - 10 = 30$$

equations, which are exact. The negative value therefore obtained for y, shows that the allowance made to the wife and son, instead of augmenting the pay of the laborer, should be regarded as a charge placed to his account. The question therefore should be modified, thus,

A laborer wrought for a person 12 days and had his wife and son with him 7 days at a certain *expense*, and received 46 shillings. He afterwards wrought 8 days, having his wife and son with him 5 days at expense as before, and received 30 shillings. How much did the laborer earn per day, and how much was charged him per day on account of his wife and son?

Resolving the question, thus stated, we have

$$x = 5, \quad y = 2.$$

From what has been done, it will be perceived, that in problems of the first degree, a negative result indicates some inconsistency in the enunciation of the question, arithmetically considered, and at the same time shows, how this inconsistency may be reconciled by rendering subtractive certain quantities, which had been regarded as additive, or additive certain quantities, which had been regarded as subtractive.

Negative results, however, in the extended sense, in which the terms addition and subtraction are used in algebra, may be regarded as answers to questions. Thus, in the equation $40 + 5x = 30$, the negative result -2 shows that it is necessary to add -10 to 40 to obtain 30. By means of this extension of the meaning of the terms, addition and subtraction, we may regard as one single question, those, the enunciations of which are such, that the solution, which satisfies one of the enunciations, will by a mere change of sign satisfy the other also.

94. The following examples will serve to exercise the learner in the interpretation of negative results.

1. A father is 55 years old, and his son is 16. In how many years will the son be one fourth as old as the father?

2. What number is that, whose fourth part exceeds its third part by 12?

3. What number is that to the numerator of which if 1 be added, its value will be $\frac{3}{5}$; but if one be added to its denominator, its value will be $\frac{5}{7}$.

4. To divide the number 30 into two such parts, that if the first be multiplied by 7 and the second by 5, the sum of the products will be 90.

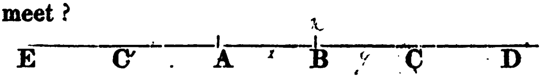
DISCUSSION OF PROBLEMS.

95. When a problem has been solved in a general manner, it may be proposed to determine what values the unknown quantities will take for particular hypotheses made upon the known quantities. The determination of these different values, and the interpretation of the results, to which we arrive, form what is called the *discussion* of the problem.

The discussion of the following problem presents nearly all the circumstances that can ever occur in equations of the first degree.

Two couriers set out at the same time from two different points A and B in the line E D and travel towards D until they meet; the courier, who sets out from the point A, travels

at the rate of m miles an hour, the other travels at the rate of n miles an hour; the distance between the points A and B is a miles; at what distance from the points A and B will they meet?



Suppose C to be the point in which they meet; let $x =$ the distance A C, $y =$ the distance B C. We have for the first equation

$$x - y = a.$$

The first courier, travelling at the rate of m miles an hour, will be $\frac{x}{m}$ hours in passing over the distance x; the second, travelling at the rate of n miles an hour, will be $\frac{y}{n}$ hours in passing over the distance y; and since these distances must each be passed over in the same time, we shall have for the second equation

$$\frac{x}{m} = \frac{y}{n}.$$

Resolving these two equations, we have

$$x = \frac{a m}{m - n}, \quad y = \frac{a n}{m - n}$$

Discussion.

1. Let m be greater than n. In this case the values of x and y will be positive, and the problem will be solved in the exact sense of the enunciation; for, it is evident, that if the courier, who sets out from A, travels faster than the other, they will meet somewhere in the direction A D.

2. Let n be greater than m. This being the case, we shall have

$$x = -\frac{a m}{n - m}, \quad y = -\frac{a n}{n - m}$$

Here the values of x and y are negative. In order to interpret this result, we observe that the courier from B travelling faster than the courier from A, the interval between them must increase continually. It is absurd therefore to require that they should meet in the direction A D. The

negative values for x and y indicate then an absurdity in the conditions of the question. To show how this absurdity may be done away, let us substitute in the equations of the problem $-x$, $-y$ instead of x and y, we shall then have

$$\left.\begin{aligned} -x+y&=a \\ -\frac{x}{m}&=-\frac{y}{n} \end{aligned}\right\} \text{ or } \left\{\begin{aligned} y-x&=a \\ \frac{x}{m}&=\frac{y}{n} \end{aligned}\right.$$

The second equation is not affected by the change of sign, as indeed it ought not to be, since it only expresses the equality of the times. In regard to the first, however, we have $y-x=a$, instead of $x-y=a$. This shows that the point, in which the couriers meet, must be nearer to A than to B by the distance A B ; it must therefore be at some point C′ on the other side of A with respect to B. In order then to remove the absurdity in the enunciation of the question, it is necessary to suppose the couriers instead of travelling in the direction A D to travel in the opposite direction B E. Indeed, if we resolve the equations

$$y-x=a$$
$$\frac{x}{m}=\frac{y}{n}$$

we have $x=\frac{am}{n-m}$, $y=\frac{an}{n-m}$ values, which are positive, and which answer the conditions of the problem modified, thus,

Two couriers set out at the same time from two points A and B in the line E D and travel towards E ; the courier, who sets out from the point B, travels at the rate of n miles an hour, the other travels at the rate of m miles an hour ; the distance between the points B and A is a miles ; at what distance from the points B and A will they meet ?

3. Let $m=n$. In this case we have $m-n=0$, and the values of x and y become

$$x=\frac{am}{0}, \quad y=\frac{an}{0}$$

But how shall we interpret this result ? Returning to the question, we perceive it to be absolutely impossible to

satisfy the enunciation; for, the couriers travelling equally fast, the interval between them must always continue the same, however far they may travel in either direction. It is impossible then that they should meet, and no change in the enunciation, so long as we have $m = n$, can make it possible. Indeed, the equations of the problem on the hypothesis $m = n$ become

$$x - y = a$$
$$x - y = 0$$

equations, which are evidently incompatible. Zero being a divisor is, then, a sign of *impossibility.*

The expressions $\frac{am}{0}$, $\frac{an}{0}$ are considered, however, by mathematicians as forming a species of value for x and y, to which they give the name of *infinite* value. To show the reason for this, let us suppose that the difference between m and n without being absolutely nothing is very small; in this case, it is evident, that the values of x and y will be very large. Let, for example, $m = 3$, $m - n = 0.01$, we shall then have $n = 2.99$, whence

$$\frac{am}{m-n} = \frac{3a}{.01} = 300\,a, \quad \frac{an}{m-n} = 299\,a$$

Again let $m - n = .0001$, m being equal to 3, n will then $= 2.9999$, whence

$$\frac{am}{m-n} = 30000\,a, \quad \frac{an}{m-n} = 29999\,a$$

In a word, so long as there is any difference, however small, between m and n, the couriers will meet in one direction or the other; but the distance of the point, in which they meet, from the points A and B will be greater in proportion as the difference between m and n is less. If then *the difference between* m *and* n *is less than any assignable quantity, the distances* $\frac{am}{m-n}$, $\frac{an}{m-n}$ *will be greater than any assignable quantity* or infinite. Since then 0 is less than any assignable quantity, we may employ this character to represent the *ultimate state* of a quantity, which may be decreased at pleas-

ure ; and since the value of a fractional quantity is greater, in proportion as its denominator is less, the expression $\frac{am}{0}$, and in general, any quantity with zero for a denominator may be considered as the symbol of an *infinite quantity*, that is, a quantity greater than any, which can be assigned.

We say then that the values $x=\frac{am}{0}$, $y=\frac{an}{0}$ are *infinite*.

To show how the notion indicated by the expression $\frac{am}{0}$ does away the absurdity of the equations

$$x-y=a, \quad x-y=0,$$

from the second of these equations, we deduce the value of y and substitute it in the first, we then have $x-x=a$; dividing both sides of this last by x, we have

$$1-1=\frac{a}{x}, \text{ or } \frac{a}{x}=0.$$

Here, as we put for x values greater and greater, the fraction $\frac{a}{x}$ will differ less and less from 0, and the equation will approach nearer and nearer to being exact. If then x be greater than any assignable quantity, $\frac{a}{x}$ will be less than any assignable quantity or zero.

4. Let us suppose next $m=n$, and at the same time $a=0$, we shall then have

$$x=\frac{0}{0}, \; y=\frac{0}{0}$$

But how shall we interpret this new result ? Returning to the enunciation, we perceive, that if the couriers set out each from the same point and travel equally fast, there is no particular point in which they can be said to meet, since in this case, they will be together through the whole extent of their route. Indeed, on this hypothesis the equations of the problem become

$$x-y=0$$
$$x-y=0$$

equations, which are identical; the problem is therefore *indeterminate*, since we have in fact but one equation with two unknown quantities. The expression $\frac{0}{0}$ is therefore a sign of *indetermination* in the enunciation of the problem.

The preceding hypotheses are the only ones, which lead to remarkable results. They are sufficient to show the manner in which algebra corresponds to all the circumstances in the enunciation of a problem.

GENERAL FORMULAS FOR EQUATIONS OF THE FIRST DEGREE WITH ONE OR TWO UNKNOWN QUANTITIES.

96. Every equation of the first degree with one unknown quantity may, by collecting all the terms, which involve x into one member and the known quantities into the other, be reduced to an equation of the form $a\,x = b$, a and b denoting any quantities whatever positive or negative.

Let there be, for example, the equation

$$\frac{x-1}{7} + \frac{23-x}{5} = 7 - \frac{4+x}{4}$$

Freeing from denominators, we have

$$20\,x - 20 + 644 - 28\,x = 980 - 140 - 35\,x$$

or transposing and uniting terms

$$27\,x = 216$$

Comparing this equation with the general formula, we have $a = 27$, $b = 216$.

Again let there be the equation

$$\frac{m\,x}{n} - p = x - q$$

Freeing from denominators transposing and uniting terms, we have $(m-n)\,x = n\,(p-q)$.

Comparing this equation with the general formula, we have $m - n = a$, $n\,(p-q) = b$.

97. Resolving the equation $a\,x = b$, we have $x = \frac{b}{a}$. This

is a general solution for equations of the first degree with one unknown quantity.

Discussion.

1. Let it be supposed, that in consequence of a particular hypothesis made upon the known quantities, we have $a = 0$, the value of x will then be $\frac{b}{0}$. But the equation $a\,x = b$ on this hypothesis becomes $0 \times x = b$, an equation which, it is evident, cannot be satisfied by any determinate value for x. The equation $0 \times x = b$ may, however, be put under the form $\frac{b}{x} = 0$. Here, if we consider x greater than any assignable quantity, the fraction $\frac{b}{x}$ will be less than any assignable quantity or zero. On this account we say that *infinity* in this case satisfies the equation. It is evident, at least, that the equation cannot be satisfied by any *finite* value for x.

2. Let us suppose next $a = 0$, and at the same time $b = 0$. the value of x will then take the form $\frac{0}{0}$. In this case the equation becomes $0 \times x = 0$, an equation which may be satisfied by any finite quantity whatever positive or negative. Thus the equation, or the problem, of which it is the algebraic translation, is *indeterminate.*

It is to be observed, however, that the symbol $\frac{0}{0}$ does not always indicate that the problem is indeterminate.

Let for example the value of x derived from the solution of a problem be

$$x = \frac{a^3 - b^3}{a^2 - b^2}$$

If we put $a = b$ in this formula, it will under its present form be reduced to $\frac{0}{0}$; but this value for x may be put under the form

$$x = \frac{(a - b)\,(a^2 + a\,b + b^2)}{(a - b)\,(a + b)}$$

If then before making the hypothesis $a = b$, we suppress the factor $a - b$, the value of x becomes

$$\frac{a^2 + a b + b^2}{a + b}$$

from which we obtain $x = \frac{3a}{2}$ on the hypothesis $a = b$.

We conclude therefore that *the symbol* $\frac{0}{0}$ *is sometimes in algebra the sign of the existence of a factor common to the two terms of a fraction, which in consequence of a particular hypothesis becomes* 0, *and reduces the fraction to this form.*

Before deciding then, that the result $\frac{0}{0}$ is a sign that the problem is indeterminate, we must examine whether the expressions for the unknown quantities, which in consequence of a particular hypothesis are reduced to this form, are in their lowest terms, if not, they must be reduced to this state; the particular hypothesis being then made anew, the result $\frac{0}{0}$ shows that the problem is really indeterminate.

98. Every equation of the first degree with two unknown quantities may be reduced to an equation of the form

$$a x + b y = c,$$

a, b and c denoting any quantities whatever positive or negative. It is evident, that all equations of the first degree with two unknown quantities may be reduced to this state 1°. by freeing the equation from denominators; 2°. by collecting into one member all the terms, which involve x and y, and the known quantities into the other; 3°. by uniting the terms, which contain x into one term, and those, which contain y into another.

Let us take the equations

$$a x + b y = c$$
$$a' x + b' y = c'$$

The letters a, b, c in the second of these equations are marked with an accent, to show that they represent quantities

different from those which are represented by the same letters in the first equation.

Resolving these equations we have

$$x=\frac{c\,b'-b\,c'}{a\,b'-b\,a'},\quad y=\frac{a\,c'-c\,a'}{a\,b'-b\,a'}$$

This is a general solution for all equations of the second degree with two unknown quantities.

To show the use, which may be made of these formulas in the solution of equations, let there be the two equations,

$$5\,x+3\,y=19,\quad 4\,x+7\,y=29$$

Comparing these with the general equations, we have

$$a=5,\, b=3,\, c=19,\, a'=4,\, b'=7,\, c'=29,$$

whence by substitution in the formulas for x and y, we have

$$x=\frac{19\times 7-3\times 29}{5\times 7-3\times 4}=\frac{133-87}{35-12}=\frac{46}{23}=2$$

$$y=\frac{5\times 29-19\times 4}{5\times 7-3\times 4}=\frac{145-76}{35-12}=\frac{69}{23}=3$$

Discussion.

In the above formulas for x and y let $a\,b'-b\,a'=0$, $c\,b'-b\,c'$ and $a\,c'-c\,a'$ being each different from zero, we shall then have

$$x=\frac{c\,b'-b\,c'}{0},\quad y=\frac{a\,c'-c\,a'}{0}$$

To interpret these results, we observe that the equation $a\,b'-b\,a'=0$ gives $a'=\frac{a\,b'}{b}$; substituting this value in the equation $a'\,x+b'\,y=c'$, we have

$$\frac{a\,b'}{b}\,x+b'\,y=c';$$

from which we obtain $a\,x+b\,y=\frac{b\,c'}{b'}$; comparing this with the equation $a\,x+b\,y=c$, the left hand members, it will be perceived, are identical, while the right are essentially different; for if in the numerator $c\,b'-b\,c'$, $c\,b'$ be

greater than $b c'$, c will be greater than $\frac{b c'}{b'}$, and if $c b'$ be less than $b c'$, c will be less than $\frac{b c'}{b}$. We conclude therefore, *that the two equations proposed cannot in this case be satisfied, at the same time, by any system whatever of finite values for x and y.* The question therefore in this case is *impossible.*

Again let us suppose $a b' - b a' = 0$, and at the same time $c b' - b c' = 0$; the value of x in this case is reduced to $\frac{0}{0}$.

To interpret this result, we remark that the equations proposed may, in consequence of the relation $a b' - b a' = 0$, be put under the form

$$a x + b y = c$$
$$a x + b y = \frac{b c'}{b'}$$

equations, which are identical, since from the relation $c b' - b c' = 0$, we have $\frac{b c'}{b'} = c$.

In order then to resolve the problem, we have in fact but one equation with two unknown quantities; the question therefore is *indeterminate.*

Since the equation $a b' - b a' = 0$ gives $b' = \frac{b a'}{a}$, we have by substitution in the equation $c b' - b c' = 0$,

$$\frac{c b a'}{a} - b c' = 0$$

or reducing, $a c' - c a' = 0$; *we infer therefore that if the value of* x *be of the form* $\frac{0}{0}$, *the value of* y *will be of the same form and the converse.*

QUESTIONS FOR SOLUTION AND DISCUSSION.

1. It is required to make a mixture of gold and silver, the weight of which shall be a grains, in such proportions that the price of the mixture shall be b shillings, the value of a grain of gold being c shillings, and that of a grain of sil-

ver d shillings; how much must be taken of each to form the mixture required?

2. A banker has two kinds of money, in the first there are a pieces to the crown, and in the second b pieces to the crown; how many pieces must be taken from each, in order that there may be c pieces to the crown?

3. It is required to divide the number a into two such parts, that if m times one part be added to n times the other part, the sum will be a given number b. What are the two parts?

4. The sides of a rectangle are to each other in the proportion of m to n, but if the quantities a and b be added to the sides respectively, the surface of the rectangle will be increased by the quantity p. What are the sides?

THEORY OF INEQUALITIES.

99. In the reasonings, which relate to the discussion of a problem, we have frequent occasion to make use of the expressions "*greater than*", "*less than*". In such cases, we shall attain a greater degree of conciseness, by representing each of these expressions by a convenient sign. It is agreed to represent the expression "*greater than*" by the sign $>$; thus, *a greater than b* is expressed by $a > b$. The same sign by a change of position is made to represent the phrase "*less than*"; thus *a less than b* is expressed by $a < b$.

An equation of the form $a = a$ is called an *equality*. An expression of the form $a > b$ or $a < b$ is called an *inequality*.

The principles established for equations apply in general to inequalities. As there are some exceptions however, we shall state the principal transformations, which may be made upon inequalities, together with the exceptions, which occur.

1°. *We may always add the same quantity to both members of an inequality, or subtract the same quantity from both members, and the inequality will continue in the same sense.*

Thus, let $3 < 5$; adding 8 to both sides, we have

$$8 + 3 < 5 + 8 \text{ or } 11 < 13.$$

Again let $-3 > -5$; adding 8 to both sides we have

$$8 - 3 > 8 - 5 \text{ or } 5 > 3.$$

This principle enables us, as in the case of equations, to transpose a term from one member of an inequality to the other; thus, from the inequality $a^2 + b^2 > 3c^2 - a^2$, we obtain $2a^2 + b^2 > 3c^2$.

2°. *We may in all cases add member to member two or more inequalities established in the same sense, and the inequality, which results, will exist in the sense of the proposed.*

Thus, let there be $a > b$, $c > d$, $e > f$; we have

$$a + c + e > b + d + f.$$

But if we subtract member from member two or more inequalities established in the same sense, the inequality, which results, will not always exist in the sense of the proposed.

Let there be the inequalities $4 < 7$, $2 < 3$, we have by subtraction $4 - 2 < 7 - 3$ or $2 < 4$.

But let there be the inequalities $9 < 10$ and $6 < 8$, subtracting the latter from the former, we have

$$9 - 6 > 10 - 8 \text{ or } 3 > 2.$$

3°. *We may multiply or divide the two members of an inequality by any positive or absolute number, and the inequality, which results, will exist in the sense of the proposed.*

Thus, if we have $a < b$, multiplying both sides by 5, we have $5a < 5b$.

By means of this principle, we may free an inequality from its denominators. Thus, let there be

$$\frac{a^2 - b^2}{2d} > \frac{a^2 - b^2}{3a},$$

we have by multiplication $(a^2 - b^2) 3a > (a^2 - b^2) 2d$, and by division $3a > 2d$.

But if we multiply or divide the two members of an inequality by a negative quantity, the inequality, which results, will exist in the contrary sense.

Thus, let $8 > 7$, multiplying both sides by -3, we have

$$-24 < -21.$$

From this it follows, that *if we change the sign of each term of an inequality, the inequality, which results, will exist in a sense contrary to that of the proposed; for this transformation will be equivalent to multiplying both members by* -1.

99. Let there now be proposed the inequality

$$7x - \frac{23}{3} > \frac{2}{3}x + 5$$

Multiplying both sides by 3, we have

$$21x - 23 > 2x + 15$$

whence, transposing and reducing, we have

$$x > 2$$

Here 2 is the *limit* to the value of x, that is, if we substitute for x in the proposed any value greater than 2 the inequality will be satisfied. The process, by which the limit to the value of the unknown quantity is determined, is called *resolving* the inequality.

100. The theory of inequalities may be applied to the solution of certain problems.

1. The double of a number diminished by 5 is greater than 25, and triple the number diminished by 7 is less than double the number increased by 13. Required a number that shall possess these properties.

By the question, we have

$$2x - 5 > 25$$
$$3x - 7 < 2x + 13$$

Resolving these inequalities, we have $x > 15$, $x < 20$. Any number therefore, entire or fractional, comprised between 15 and 20 will satisfy the conditions of the question.

2. A shepherd being asked the number of his sheep replied, that double their number diminished by 7 is greater than 29, and triple their number diminished by 5 is less than double their number increased by 16. Required the number of sheep.

Resolving the question, we have $x > 18$, and $x < 21$. Here all the numbers, comprised between 18 and 21, will satisfy the inequalities; but since the nature of the question

requires that the answer should be an entire number, the number of solutions is limited to 2, viz. $x = 19$, $x = 20$.

3. A market woman, has a number of oranges, such, that triple the number increased by 2, exceeds double the number increased by 61; and 5 times the number diminished by 70 is less than 4 times the number diminished by 9. How many oranges had she?

INDETERMINATE ANALYSIS OF THE FIRST DEGREE.

We have seen art. 85, that when the conditions of a problem furnish fewer equations, than there are unknown quantities to be determined, the equations of the problem admit of an infinity of systems of values for the unknown quantities. It is frequently the case, however, that the nature of the question requires that the values of the unknown quantities should be entire numbers. By this circumstance the number of solutions, it is easy to see, will be much restricted, especially, if we reckon only the *direct* solutions, that is to say, solutions in *entire and positive* numbers.

Let it be proposed to find two numbers, such, that the first added to three times the second shall be equal to 15.

Putting x and y for the numbers sought, we have by the question

$$x + 3y = 15$$

whence

$$x = 15 - 3y$$

Here, if by x and y we understand any numbers whatever entire and fractional, positive or negative, the question it is easy to see, admits of an infinite number of solutions. But if the nature of the question requires, that x and y should be entire and positive numbers, the number of solutions, it is evident, will be much restricted by this circumstance. Indeed, in order that x and y may be positive, the value of y, it is evident, must not exceed 5. If then we put successively for y

$$y = 0,\ 1,\ 2,\ 3,\ 4,\ 5$$

we have

$$x = 15,\ 12,\ 9,\ 6,\ 3,\ 0$$

and the question admits of six different solutions only, the solution in which 0 is reckoned as a value of one of the unknown quantities being included.

Problems of the kind, which we are here considering, are called *indeterminate problems*, and that part of algebra, which relates to the solution of indeterminate problems, is called *indeterminate analysis*.

101. The preceding question, in which the coefficient of one of the unknown quantities is equal to unity, presents no difficulty. We shall now show, that whatever the coefficients of the unknown quantities may be, the solution of the question proposed may be made to depend upon the resolution of an equation, in which the coefficient of one of the unknown quantities is equal to unity.

Let it be proposed then to find the entire values of x and y in the equation $17\,x + 11\,y = 542$

Deducing from this equation the value of y, we have

$$y = \frac{542 - 17\,x}{11}$$

or performing the division as far as possible, we have

$$y = 49 - x + \frac{3 - 6\,x}{11}$$

Here, since by the question the values of x and y should be entire numbers, *it is necessary and it is sufficient*, that $\frac{3 - 6\,x}{11}$ should be equal to an entire number. Let t be this number (t is called an *indeterminate*), we have

$$y = 49 - x + t$$

$$t = \frac{3 - 6\,x}{11}$$

deducing from this last the value of x, we have

$$x = \frac{3 - 11\,t}{6}$$

or performing the division as far as possible

$$x = -t + \frac{3 - 5\,t}{6}.$$

Here, since x and t are entire numbers, $\frac{3 - 5\,t}{6}$ must be

equal to an entire number ; let t' be this number, we have

$$x = -t + t'$$
$$t' = \frac{3 - 5t}{6}$$

deducing from this last equation the value of t, we have

$$t = -t' + \frac{3 - t'}{5}$$

but t and t' in this equation are entire numbers ; $\frac{3 - t'}{5}$ must therefore be equal to an entire number ; let t'' be this number, we have

$$t = -t' + t''$$
$$t'' = \frac{3 - t'}{5}$$

deducing from this last the value of t', we have

$$t' = 3 - 5t''$$

Here the operation terminates, since, it is evident, that if we put any entire number whatever for t'', we shall have an entire value for t' as is required. The solution of the proposed moreover depends upon this last equation, in which the coefficient of one of the unknown quantities t' is equal to unity. Indeed, the two principal unknown quantities and the intermediate indeterminates are, it is evident, connected together by the equations

$$y = 49 - x + t$$
$$x = -t + t'$$
$$t = -t' + t''$$
$$t' = 3 - 5t''$$

if then we give any entire value whatever to t'', and return from the last of these equations to the first two, we shall obtain entire values for x and y, which will necessarily satisfy the equation proposed ; since, as is evident from the course which has been pursued, the equation proposed will be reproduced by the elimination of t, t', t'' in the four equations established above.

To determine with more facility the values of t'', which will give entire values for x and y, we express x and y im-

mediately in terms of t''; performing the operations necessary for this purpose, we have

$$x = 6 - 11\,t''$$
$$y = 40 + 17\,t''$$

These, by the elimination of t'', reproduce the equation proposed. If then we make successively $t'' = 0, 1\ 2, 3, \ldots$ or otherwise, $t'' = 0, -1, -2, -3, \ldots$ in the above, we shall obtain all the entire values of x and y proper to satisfy the equation proposed. But if entire and positive values only are required for x and y, it will be necessary to give to t'' such values only, as will render $6 - 11\,t''$, $40 + 17\,t''$ positive. It is evident, that $t'' = 0$, $t'' = -1$, $t'' = -2$ are the only values of t'', that will fulfil this condition; for, every positive value of t'' will render x negative, and every negative value of t'' numerically greater than 2 will render y negative. Putting therefore $t'' = 0, -1, -2$ successively, we have

$$x = 6,\ 17,\ 28$$
$$y = 40,\ 23,\ 6$$

The proposed therefore admits of three different solutions in entire numbers, and of three only.

Let it be proposed, as a second example, to divide the number 159 into two such parts, that the first may be divisible by 8, and the second by 13.

Designating by x and y the quotients, arising from the division of the parts sought by the numbers 8 and 13 respectively, we have by the question $8\,x + 13\,y = 159$.

Pursuing with this equation the same process, as in the preceding example, we have the five equations

$$\begin{aligned} x &= 19 - y + t \\ y &= \ 1 - t + t' \\ t &= \quad\ - t' + t'' \\ t' &= 1 - t'' - t''' \\ t'' &= 2\,t''' \end{aligned}$$

Expressing x and y in terms of t''', we have

$$x = 15 + 13\,t'''$$
$$y = 3 - 8\,t'''$$

Here it is evident, that $t''' = 0$, and $t''' = -1$ are the only values of t''', which will give entire and positive values for x and y. Making successively $t''' = 0$, $t''' = -1$, we have

$$x = 15,\ 2$$
$$y = 3,\ 11$$

Since then $8\,x$ and $13\,y$ represent the parts required, the proposed admits of two solutions, viz. 120 and 39 for the first solution, and 16 and 143 for the second.

102. Let it be required, as a third example, to solve in entire numbers the equation $49\,x - 35\,y = 11$.

Here, it will be observed, that the coefficients of x and y have a common factor 7; dividing therefore both members by 7, we have $7\,x - 5\,y = \frac{11}{7}$, an equation which is evidently impossible in entire numbers; the proposed therefore does not admit of entire and positive values for x and y. In general, the proposed equation being reduced to the form $a\,x + b\,y = c$, *if the coefficients of* x *and* y *have a common factor, which does not enter into the second member, the equation is impossible in entire numbers.*

If there be a factor, common to the coefficients of x and y, which does not enter into the second member, and this factor be not perceived at first, the course of the calculation will make known sooner or later the impossibility of solving the question in entire numbers.

Applying the process explained above to the equation $49\,x - 35\,y = 11$, we obtain finally the equation

$$t = 2\,t' - 1 - \frac{4}{7},$$

an equation, which is evidently impossible in entire numbers for t and and t', from which it is readily inferred, that the proposed will not admit of entire solutions.

If the equation of the proposed question have therefore a factor common to both members, we suppress it; the coefficients of x and y will then be prime to each other, if the question admits of solution in entire and positive numbers. This being the case, the process explained above will always

lead to a final equation, in which the coefficient of one of the indeterminates is equal to unity. Indeed, it will readily be perceived, that in the course pursued we apply to the coefficients of x and y in the proposed the process of the greatest common divisor; since then these coefficients are by hypothesis prime to each other, we arrive finally at a remainder equal to unity, which will be the coefficient of the last but one of the *indeterminates* introduced in the course of the calculation.

103. In certain cases the preceding process admits of simplifications, which it is important to introduce in practice.

Let it be required, for example, to resolve in entire numbers the equation $17x - 49y = -8$.

Deducing from this equation the value of x, we have

$$x = \frac{49y - 8}{17}$$

or performing the division as far as possible, we have

$$x = 2y + \frac{15y - 8}{17}.$$

but 49, it will be observed, $= 17 \times 3 - 2$, whence

$$\frac{49y}{17} = 3y - \frac{2y}{17};$$

the value of x may therefore take the form

$$x = 3y - \frac{2y + 8}{17} = 3y - \frac{2(y + 4)}{17}$$

Thus, in order that x and y may be entire numbers, it is necessary; that $\frac{2(y + 4)}{17}$ should be equal to an entire number, but since 2 and 17 are prime to each other, in order that $\frac{2(y + 4)}{17}$ may be equal to an entire number, it is necessary that $\frac{y + 4}{17}$ should be equal to an entire number; putting t for this number, we have

$$x = 3y - 2t$$

$$t = \frac{y + 4}{17}$$

From which we obtain

$$y = 17\,t - 4$$
$$x = 49\,t - 12$$

Here, every entire and positive value for t will give similar values for x and y; but if we suppose $t = 0$, or to be negative, the values of x and y will be negative. The number of entire and positive solutions of the proposed is therefore *infinite*, and the smallest system of values for x and y is

$$x = 37,\ y = 13.$$

By means of the change of form in the value of x, introduced in the preceding solution, we employ one indeterminate only in the course of the calculation; whereas without this modification three will be necessary.

The same simplification may be introduced in the solution of the following question.

To find a number, which being divided by 39 gives a remainder 16, and divided by 56 gives a remainder 27.

Let $x =$ the entire quotient arising from the division of the number sought by 39, and $y =$ the quotient arising from the division of this number by 56; we have by the question

$$39\,x + 16 = 56\,y + 27$$

or reducing $$39\,x - 56\,y = 11$$

whence $$x = \frac{56\,y + 11}{39} = y + \frac{17\,y + 11}{39}$$

or otherwise $$x = 2\,y - \frac{(22\,y - 11)}{39} = 2\,y - \frac{11\,(2\,y - 1)}{39}$$

Thus, in order that x and y may be entire, it is necessary that $\frac{2\,y - 1}{39}$ should be equal to an entire number; let t be this number, we have

$$x = 2\,y - 11\,t$$
$$t = \frac{2\,y - 1}{39}$$

whence we obtain finally for x and y

$$x = 56\,t' - 27$$
$$y = 39\,t' - 19$$

From inspection of these formulas we perceive that t' may

have any positive value whatever. Let $t' = 1$, we have $y = 20$, $x = 29$; substituting this value of x in the expression $39x + 16$, we have 1147 for the smallest number, that will satisfy the enunciation of the question.

104. From what has been done, it will be perceived, that if the equation proposed be of the form $ax + by = c$, the number of solutions in entire and positive numbers will be *limited*; but if the equation be of the form $ax - by = c$, c being either positive or negative, the number of solutions will be *infinite*.

If moreover we compare the formulas for x and y with the equations from which they are derived, the coefficient of the indeterminate in the formula for x is the same, it will be observed, with the coefficient of y in the equation; and the coefficient of the indeterminate in the formula for y is equal to the coefficient of x in the equation, taken with the contrary sign, or the converse, as it respects the signs of the coefficients. Having obtained then a first solution of the question proposed, those which follow will be found by adding successively to the values of x the coefficient of y in the equation, and subtracting successively from the values of y the coefficient of x in the equation or the converse, the coefficients of x and y being taken with the signs, which they have in the equation.

EXAMPLES.

1. It is required to divide 25 into two parts, one of which may be divisible by 2, and the other by 3.

2. To divide 100 into two such parts, that the one may be divisible by 7, and the other by 11.

3. To divide 100 into two such parts, that if the first be divided by 5 the remainder will be 2; and if the second be divided by 7 the remainder will be 4.

4. To find a number such, that 9 times the first, diminished by 7 times the second, may be equal to 50.

5. A person buys some horses and oxen; he pays $31

for each horse, and \$20 for each ox; and he finds that the oxen cost him \$7 more than the horses. How many horses and oxen did he buy?

105. We pass next to the solution of problems and equations with three or more unknown quantities.

1. Let it be proposed to pay 741 livres with 41 pieces of money of three different species, viz. pieces of 24 livres, 19 livres, and 10 livres.

Let x, y and z represent respectively the number of pieces of each kind, we have by the question

$$x + y + z = 41$$
$$24\,x + 19\,y + 10\,z = 741$$

Eliminating one of the unknown quantities x, for example, we have $5\,y + 14\,z = 243$

Deducing from this equation formulas for entire values for z and y, according to the method explained above, we have

$$z = 5\,t' - 3$$
$$y = 57 - 14\,t'$$

Substituting next, in the first of the equations of the proposed, the expressions for z and y just obtained, and deducing the value of x, we have

$$x = 9\,t' - 13$$

If we now put for t', in the above formulas for x, y, and z, any entire values whatever, we shall obtain entire values for x, y, and z, which will satisfy the equations of the proposed. But to obtain the entire and positive values only, as the nature of the question requires, it is evident, 1°. that $9\,t'$ should be greater than 13, or which is the same thing, that t' should be greater than $1\frac{4}{13}$; 2°. that $14\,t'$ should be less than 57, or which is the same thing, that t' should be less than $4\frac{1}{14}$; t' can therefore have only the values 2, 3, 4.

Putting in the formulas above t' equal to 2, 3, and 4 successively, we have

$$x = 5,\ 14,\ 23$$
$$y = 29,\ 15,\ 1$$
$$z = 7,\ 12,\ 17$$

The proposed therefore admits of three different solutions and of three only.

Let us take, as a second example, the equations

$$6x + 7y + 4z = 122$$
$$11x + 8y - 6z = 145$$

Eliminating z from these equations, we have

$$40x + 37y = 656$$

Deducing from this equation formulas for entire values for x and y, we have

$$x = 37t + 9$$
$$y = 8 - 40t$$

Substituting these expressions for x and y in the first of the proposed equations, and reducing, we have

$$2z - 29t = 6$$

Deducing from this last, formulas for entire values for z and t, we have

$$z = 29t' + 3$$
$$t = 2t'$$

Substituting next in the formulas for x and y, obtained above, $2t'$ for t, the formulas for x, y, and z will be

$$x = 74t' + 9$$
$$y = 8 - 80t'$$
$$z = 29t' + 3$$

Putting any entire values whatever for t' in these formulas, we obtain entire values for x, y, and z, which will satisfy the equations proposed. But if entire and positive values are required, it is evident, that t' cannot be positive, for then y will be negative; neither can t' be negative, for then x and z will be negative. The hypothesis $t' = 0$, however, will give $x = 9$, $y = 8$, $z = 3$. The proposed equations therefore admit of but one direct solution.

From what has been done, it will be easy to see how we are to proceed in the case of three equations with four unknown quantities and so on.

EXAMPLES.

1. Thirty persons, men, women, and children, spend 50 crowns in a tavern. The share of a man is 3 crowns, that of a woman 2 crowns, and that of a child is 1 crown. How many persons were there of each class ?

2. A coiner has three kinds of silver, the first of 7 ounces, the second of $5\frac{1}{2}$ ounces, the third of $4\frac{1}{2}$ ounces, fine per marc of eight ounces. How many marcs must he take of each sort, in order to form a mixture of the weight of 30 marcs, at 6 ounces fine per marc ?

3. To find three entire numbers such, that if the first be multiplied by 3, the second by 5, and the third by 7, the sum of the products will be 560; and if the first be multiplied by 9, the second by 25, and the third by 49, the sum of the products will be 2920.

106. In the preceding examples, the number of equations has been one less than the number of unknown quantities to be determined. It is sometimes the case that the number of equations is less by two or more units, than the number of unknown quantities; the problem is then said to be *indeterminately indeterminate* or *more than indeterminate.*

The solution of the following problem will exhibit the course to be pursued in cases of this kind.

It is required to pay a debt of 187 francs with pieces of 5 francs, 6 francs, and 20 francs, without any other coin.

Let x, y, and z be the number of pieces of each sort respectively; we have by the question

$$5x + 6y + 20z = 187$$

which returns to $$5x + 6y = 187 - 20z$$

Let $187 - 20z = c$, we have $5x + 6y = c$, from which we obtain $$x = \frac{c - 6y}{5} = -y + \frac{c - y}{5}$$

let $\frac{c-y}{5} = t$; deducing the value of y, we have $y = c - 5t$

and by consequence $$x = -c + 6t$$

or restoring the value of c, we have finally

$$x = -187 + 20z + 6t$$
$$y = 187 - 20z - 5t$$

Here in order that x and y may be entire, it will be sufficient to give to z and t any entire values whatever ; but if, as the nature of the question requires, we seek the entire and positive values of the unknown quantities, it is evident from the equation $5x + 6y + 20z = 187$, that z cannot receive values greater than $\frac{187}{20}$ or $9\frac{7}{20}$, otherwise x or y would be negative.

Let us then put successively $z = 0, 1, 2, \ldots 9$. Making $z = 0$ the values of x and y become

$$x = -187 + 6t$$
$$y = 187 - 5t$$

Here, in order that x and y may be positive, it is necessary that t should be greater than $31\frac{1}{6}$, but less than $37\frac{2}{5}$. Putting therefore successively $t = 32, 33, \ldots 37$, we have for the hypothesis $z = 0$

$$x = 5, 11, 17, 23, 29, 35$$
$$y = 27, 22, 17, 12, 7, 2$$

Putting $z = 1$, the formulas for x and y become

$$x = -167 + 6t$$
$$y = 167 - 5t$$

Here, in order that x and y may be positive, t should be greater than $27\frac{5}{6}$, but less than $33\frac{2}{5}$. Putting $t = 28, \ldots 33$ successively, we have for the hypothesis $z = 1$,

$$x = 1, 7, 13, 19, 25, 31$$
$$y = 27, 22, 17, 12, 7, 2$$

Making $z = 8$, the formulas for x and y become

$$x = -27 + 6t$$
$$y = 27 - 5t$$

Here, in order that x and y may be positive, t should be greater than $4\frac{1}{2}$, but less than $5\frac{2}{5}$; whence for the hypothesis $z = 8$, we have $x = 3$, $y = 2$; thus the proposed admits of but one solution for this hypothesis.

If we put $z = 9$, the formulas for x and y become,

$$x = -7 + 6t, \quad y = 7 - 5t;$$

x and y therefore can have no entire and positive values for this hypothesis.

MISCELLANEOUS EXAMPLES.

1. To pay a debt of 78 francs with pieces of 5 francs and of 3 francs, without any other coin.

2. A company of men and women spend 1000 shillings at a tavern. The men paid each 19 shillings, and each woman 13. How many men and women were there?

3. To divide 1000 into two such parts, that the first may be divisible by 7, and the second by 13.

4. A company of men and women club together for the payment of a reckoning ; each man pays 25 shillings, and each woman 16 shillings, and it is found that all the women together have paid 1 shilling more than the men. How many men and women were there ?

5. A coiner has gold of 15, of 17, and of 22 carats fine. How many ounces must he take of each, in order to form a mixture of the weight of 35 ounces, 20 carats fine.

6. To find a number such, that divided by 11 the remainder will be 3, divided by 19 the remainder will be 5, and divided by 29, the remainder will be 10.

7. A person buys 100 head of cattle for 100 crowns, viz. oxen at 10 crowns each, cows at 5 crowns, calves at 2 crowns, and sheep at $\frac{1}{2}$ crown each. How many of each kind did he buy ?

8. It is required to pay a debt of 139 francs, by means of pieces of 3 francs, 5 francs, and 20 francs, without any other coin.

9. To find three numbers such, that if the first be divided by 3 the remainder will be 2, if the second be divided by 6 the remainder will be 3, and if three times the first be diminished by twice the second, the remainder will be 21.

10. Three bodies set out at the same time from a given point in the circumference of a circle, and move in the same direction with the velocities v, v', v'' in an hour respectively.

It is required to determine the times, in which the bodies will meet two and two, three and three, the length of the circumference being denoted by c.

Let us suppose $v' > v$ and $v'' > v'$. The spaces passed over by the bodies in an hour will be as v, v', v'' respectively, and in order that any two of them may meet, it is necessary and it is sufficient, that the space passed over by one should be equal to the space passed over by the other in the same time increased by an exact number of circumferences. This being the case, let it be supposed that after x hours the first two bodies are upon the same point of the circumference, the spaces passed over by each will be $v\,x$, $v'\,x$ respectively, and n representing any entire and positive number whatever, we shall have

$$v'\,x = v\,x + n\,c$$

$$x = \frac{n\,c}{v' - v}$$

Putting in this formula $n = 1, 2, 3$, &c. successively, we shall have the number of hours required for the second body to meet the first, for the first, second, third,&c. times.

In like manner, let it be supposed that after y hours the third and first bodies are upon the same point of the circumference, and that after z hours the third and second are upon the same point of the circumference, n' and n'' denoting each any entire and positive numbers whatever, we shall have

$$y = \frac{n'\,c}{v'' - v}, \quad z = \frac{n''\,c}{v'' - v'}$$

To determine therefore, the times in which the bodies will meet two and two, it will be sufficient, to give to the indeterminates n, n', n'' in the above formulas, the values 1, 2, 3, &c. successively.

To determine next, the times, in which the bodies will meet three and three. Since the second body will meet the first after x hours, and the third will meet the first after y hours, in order that the three bodies may meet, it is neces-

sary and it is sufficient, that the values of x and y should be equal; we have then

$$\frac{n\,c}{v'-v}=\frac{n'\,c}{v''-v} \text{ or, } \frac{n}{n'}=\frac{v'-v}{v''-v}$$

and the question is reduced to determine the entire and positive values of n and n', which will satisfy this last equation. Let $\frac{v'-v}{v''-v}$, reduced to its most simple expression, $=\frac{a'}{b'}$, we shall then have $n=a'$, $n'=b'$, and all the entire and positive values of n and n' required will be deduced from the formulas $n=a'\,e$, $n'=b'\,e$ by making e equal successively to the numbers 1, 2, 3, &c. Then since the number of hours elapsed from the departure of the three bodies until they meet, will be expressed by either of the formulas

$$\frac{n\,c}{v'-v},\ \frac{n'\,c}{v''-v},$$

denoting this number of hours by t, we shall have

$$t=\frac{a'\,e\,c}{v'-v} \text{ or } t=\frac{b'e\,c}{v''-v}$$

Making in either of these formulas the indeterminate e equal 1, 2, 3, &c. successively, we shall have the times, in which the bodies will meet three and three.

The following are particular cases of this problem.

1. The hour, minute, and second hands of a watch are together at 12 o'clock. When will they be together two and two, three and three?

2. There is an island 73 miles in circumference, and 3 footmen all start together to travel the same way about it; A goes 5 miles a day, B 8, and C 10; when will they all come together again?

EQUATIONS OF THE SECOND DEGREE.

107. Let it now be proposed to find a number, which multiplied by five times itself will give a product equal to 125.

Putting x for the number required, we have by the ques-

tion $5x^2 = 125$, from which we obtain $x^2 = 25$. This equation is essentially different from any, which we have hitherto considered. It is called an equation of the *second degree*, because it contains x raised to the second power. To find the value of x, we must see what number multiplied by itself will give 25. It is obvious, that the number 5 will fulfil this condition; whence we have $x = 5$.

The value of x is easily found in the present example, but in others it will be more difficult. Hence arises this new arithmetical question, viz. *To find a number, which multiplied by itself will give a product equal to a proposed number*, or which is the same thing, from the second power of a number to determine the first.

A number, which multiplied by itself will produce a given number, is called the *square* or *second root* of this number. The process for finding the second root is called *extracting* the square or second root.

In the following table, we have the nine primitive numbers with their squares written under them respectively.

1,	2,	3,	4,	5,	6,	7,	8,	9,
1,	4,	9,	16,	25,	36,	49,	64,	81.

By inspection of this table, it will be perceived, that among entire numbers consisting of one or two figures, there are nine only, which are squares of other entire numbers. The remainder have for a root an entire number plus a fraction. Thus 53, which is comprised between 49 and 64, has for its square root 7 plus a fraction.

The numbers in the second line of this table being the squares of those in the first, conversely the numbers in the first line are the square roots of those in the second. If therefore the number, the square root of which is required, consist of one or two figures only, its root will be readily found by means of the table.

Let it be proposed to find the root of a number consisting of more than two figures, 6084, for example.

The square of 9, the largest number consisting of one

figure, is 81, and the square of 100, the smallest number consisting of three figures, is 10000 ; the square root of 6084 will therefore consist of two places, viz. units and tens.

To determine then a method, by which to return from the proposed number to its root, let us observe the manner, in which the different parts of a number consisting of two places, 47, for example, are employed in forming the square of this number. For this purpose we decompose 47 into two parts, viz. 40 and 7, or 4 tens and 7 units. Designating the tens by a and the units by b, we have $a+b=47$, and squaring both sides $a^2+2\,ab+b^2=2209$. Thus the square of a number, consisting of units and tens, is composed of three parts, viz. the *square of the tens, plus twice the product of the tens multiplied by the units, plus the square of the units.* Thus in 2209, the square of 47, we have

The square of the tens (a^2)	=	1600
twice the tens by the units ($2\,ab$)	=	560
the square of the units (b^2)	=	49
		2209

Considering then the proposed number 6084 as composed of the square of the tens of the root sought, twice the product of the tens by the units, and the square of the units, if we can discover in this number the first of these parts, viz. the square of the tens, the tens of the root will be readily found. The square of an exact number of tens, it is evident, can have no figure inferior to hundreds. Separating then the two last figures of the proposed from the rest by a comma, the square of the tens will be found in 60, the part at the left of the comma, which in addition to the hundreds in the square of the tens will also contain those, which arise from the other parts of the square. 60 is comprised between 49 and 64, the roots of which are 7 and 8 respectively ; 7 will then be the figure denoting the tens in the root sought. Indeed 60 00 is comprised between 49 00 and 64 00, the squares of 70 and 80 respectively ; the same is the case with 60 84 ; the root required will therefore consist of 7 tens and a certain number of units less than ten.

The figure 7 being thus obtained, we place it at the right of the proposed, taking care to separate them by a vertical line; we then subtract 49, the square of 7, from 60, and to the remainder 11 we bring down 84, the two other figures of the proposed. The result 1184 of this operation will then contain twice the product of the tens of the root by the units, plus the square of the units. Twice the product of the tens by the units will, it is evident, contain no figure inferior to tens. Separating then 4, the left hand figure of the remainder 1184, from the rest by a comma, the part 118 of this remainder, at the left of the comma, must contain the double product of the tens by the units, together with the tens arising from the square of the units.

The double product of the tens is 14; dividing therefore 118 by 14, the quotient 8 will be the unit figure exactly, or in consequence of the tens arising from the square of the units, it may be too large by 1 or 2. To determine whether 8 be the right figure for the units of the root, we multiply twice the tens by 8 and subtract the result from 1184, the remainder 64 being equal to the square of 8 shows that 8 is the unit figure sought. We have 78 therefore for the root required. The operation will stand thus,

$$\begin{array}{r|l} 60,84 & 7 \\ 49 & \\ \hline 118,4 & \frac{14}{8} \\ 112 & \\ \hline 64 & \\ 64 & \end{array}$$

To complete the root, we place 8 the unit figure at the right of 7 the figure for the tens. The work, moreover, may be abridged by writing the 8 at the right of the divisor, and multiplying 148 the number thus formed by 8. We thus obtain in one expression twice the tens by the units and the square of the units; this being equal to the remainder 1184 proves as before that 8 is the right figure for the units of the root.

With this modification, the work will stand thus

$$\begin{array}{l|l} 60,84 & 78 \\ 49 & \\ \hline 118,4 & 148 \\ 1184 & \\ \hline \end{array}$$

Let us take, as a second example, the number 841. Pursuing the same course as in the preceding example, we find 2 for the tens of the root; subtracting the square of the tens the remainder will be 441. Separating the unit figure in this remainder from the rest by a comma, and dividing the part at the left by double the tens, in order to obtain the unit figure of the root, we have 11 for the result. This is evidently too much. Indeed, we cannot have more than 9 for the units; we therefore try 9. This proves to be the correct figure. The root sought is therefore 29.

The operation will be as follows

$$\begin{array}{l|l} 8,41 & 29 \\ 4 & \\ \hline 44,1 & 49 \\ 44\ 1 & \\ \hline \end{array}$$

108. Any number however large may be considered as composed of units and tens; 345, for example, may be considered as composed of 34 tens and 5 units.

Let it now be proposed to find the second root of 190969. This number exceeds 10 000 and is less than 1000 000; its root will therefore consist of three places. But from what has been said, the root may be considered as composed of two parts, units and tens. The proposed will then consist of three parts, viz. the square of the tens of the root, twice the tens by the units and the square of the units. The square of the tens will have no figure inferior to hundreds. Separating therefore the last two figures from the rest by a comma, the tens of the root will be found by extracting the square root of 1909, the part of the proposed at the left of the com-

ma. Regarding 1909 for the moment as a separate number, its root will evidently consist of two places, units and tens. The method of finding the root will therefore be the same as in the preceding examples. Performing the necessary operations we obtain 43 for the root and a remainder of 60. There will therefore be 43 tens in the root of the proposed, and bringing down the last two figures of the proposed by the side of 60, the result 6069 will contain twice the product of the tens of the root sought by the units, plus the square of the units. Separating therefore the right hand figure from the rest by a comma, we divide 606, the part on the left of the comma, by 86 twice the tens; this gives 7 for the unit figure. Placing the 7 therefore at the right of 43 the part of the root already found, and also at the right of 86, and multiplying this last by 7, we have 6069 for the result. 7 is therefore the right unit figure, and the root of the proposed is 437.

The following is a table of the operations.

```
19,09,69 | 437
16       |
--------
  309 | 867
  249
  ------
   606,9
   606,9
   ------
```

The same process, it is easy to see, may be extended to any number however large. From what has been done therefore, the following rule for the extraction of the second root will be readily inferred, viz. 1°. *Separate the number into parts of two figures each, beginning at the right.* 2°. *Find the greatest second power in the left hand part; write the root as a quotient in division, and subtract the second power from the left hand part.* 3°. *Bring down the two next figures at the right of the remainder for a dividend and double the root already found for a divisor. See how many times the divisor is contained in the dividend, neglecting the right hand figure. Write the re-*

sult in the root at the right of the figure previously found, and also at the right of the divisor. 4°. *Multiply the divisor, thus augmented, by the last figure of the root and subtract the product from the whole dividend.* 5°. *Bring down the next two figures as before, to form a new dividend, and double the root already found for a divisor, and proceed as before. The root will be doubled, if the right hand figure of the last divisor be doubled.*

109. If the number proposed be not a perfect square, we shall obtain by the above rule, the root of the greatest square number contained in the proposed. Thus, let it be required to find the square root of 1287. Applying the rule to this number, we obtain 35 for the root with a remainder 62. This remainder shows that 1287 is not a perfect square. The square of 35 is 1225, that of 36 is 1296 ; whence 35 is the root of the greatest square contained in the proposed.

110. When the proposed number is not a perfect square a doubt may sometimes arise, whether the root found be that of the greatest square contained in this nnmber. This may be readily determined by the following rule. The square of $a + 1$ is $a^2 + 2a + 1$; whence the square of a quantity greater by unity than a exceeds the square of a by $2a + 1$. From this it follows, that *if the root obtained should be augmented by unity or more than unity, the remainder after the operation must be at least equal to twice the root plus unity.* When this is not the case, the root obtained is that of the greatest square contained in the proposed.

EXAMPLES.

1. To find the square root of 56821444.
2. To find the square root of 17698849.
3. To find the square root of 698485673.

111. From what has been done, it will be perceived, that there are many whole numbers, the roots of which are not whole numbers. What is remarkable in regard to these numbers is, that they will have no assignable roots. Thus the numbers 3, 7, 11 have no assignable roots, that is, no

number can be found either among whole or fractional numbers, which multiplied by itself will produce either of these numbers. The proof of this depends upon the following proposition, which we shall now demonstrate, viz.

Every number P, *which will exactly divide the product* A B *of two numbers* A *and* B, *and which is prime to one of these numbers, must necessarily divide the other number.*

Let us suppose that P will not divide A, and that A is greater than P. Let us apply to A and P the process of the greatest common divisor, designating the quotients, which arise by $Q, Q', Q'' \ldots$ and the remainders by $R, R', R'' \ldots$ respectively. It is evident, that if the operation be pursued sufficiently far, we shall obtain a remainder equal to unity, since by hypothesis A and P are prime to each other. This being premised, we have the following equations

$$A = P\,Q + R$$
$$P = R\,Q' + R'$$
$$R = R'\,Q'' + R''$$
$$\ldots\ldots\ldots$$
$$\ldots\ldots\ldots$$

Multiplying the first of these equations by B, and dividing by P, we have

$$\frac{A\,B}{P} = B\,Q + \frac{B\,R}{P}$$

By hypothesis $\frac{A\,B}{P}$ is an entire number, and since B and Q are each entire numbers, the product B Q is an entire number. It follows therefore, that $\frac{B\,R}{P}$ must be an entire number; whence B multiplied by the remainder R is divisible by P.

Again, multiplying the second of the above equations by B and dividing by P, we have

$$B = \frac{B\,R\,Q'}{P} + \frac{B\,R'}{P}$$

But we have already shown, that $\frac{B\,R}{P}$ is an entire num-

ber; whence $\frac{B\,R\,Q'}{P}$ is an entire number. This being the case, $\frac{B\,R'}{P}$ must be an entire number; whence B multiplied by the remainder R′ must be divisible by P.

If then the remainder R′ is equal to unity, the proposition is demonstrated, since in this case we shall have $B \times 1$ or B divisible by P. But if the remainder R′ is not equal to unity, it is evident, that if the process of the greatest common divisor be applied to the quantities A and P until a remainder is obtained equal to unity, we may, in the same manner as above, prove that B multiplied by this remainder will be divisible by P.

We conclude therefore that if P, which we have supposed not to divide A, will not divide B, it will not divide A B the product of A by B.

Returning now to our purpose, it is evident, in order that a fractional number $\frac{a}{b}$ may be the root of an entire number c, we must have

$$\frac{a^2}{b^2} = c$$

But if c be not a perfect square, its root will not be an entire number, that is, a will not be divisible by b; but from what has just been demonstrated, if a is not divisible by b, $a \times a$ or a^2 will not be divisible by b, and by consequence a^2 will not be divisible by b^2; whence $\frac{a^2}{b^2}$ cannot be equal to an entire number c.

112. Though the roots of numbers, which are not perfect squares cannot be assigned either among whole or fractional numbers, yet, it is evident, there must be a quantity, which multiplied by itself will produce any number whatever. Thus the root of 53 cannot be assigned; yet there must be a quantity, which multiplied by itself will produce 53. This quantity, it is evident, lies between the numbers 7 and 8, for the square of 7 is 49, and the square of 8 is 64. If

then we divide the difference between 7 and 8 by means of fractions, we shall obtain numbers, the squares of which will be greater than 49 and less than 64, and which will approach nearer and nearer to 53.

113. All numbers whether entire or fractional have a common measure with unity; on this account they are said to be *commensurable*; and since the ratio of these numbers to unity may always be expressed by entire numbers, they are on this account called *rational* numbers.

The root of a number which is not a perfect square, can have no common measure with unity; for, since it is impossible to express this root by any fraction into how many parts soever we conceive unity to be divided, no fraction can be assigned sufficiently small to measure at the same time this root and unity. The roots of numbers, which are not perfect squares, are on this account called *incommensurable* or *irrational* quantities. They are sometimes also called *surds*.

To indicate that the square root of a quantity is to be taken, we use the character $\sqrt{}$, which is called a radical sign. Thus $\sqrt{16}$ is equivalent to 4. $\sqrt{2}$ is an *incommensurable* or *surd* quantity.

EXTRACTION OF THE SECOND ROOT OF FRACTIONS.

114. Since a fraction is raised to the second power by raising the numerator to the second power, and the denominator to the second power, it follows, that the second root of a fraction will be found by extracting the second root of the numerator, and of the denominator. Thus, the second root of $\frac{9}{16}$ is $\frac{3}{4}$.

If either the numerator or denominator of the fraction is not a perfect square, the root of the fraction cannot be found exactly. We may, however, always render the denominator of a fraction a perfect square by multiplying both terms of the fraction by the denominator. This will not alter the value of the fraction. The root of the denominator may then be found, and for that of the numerator, we must take the

number nearest the root. Thus, if it be required to extract the square root of $\frac{3}{5}$, multiplying both terms by 5, the fraction becomes $\frac{15}{25}$, the root of which is nearest $\frac{4}{5}$, accurate to within $\frac{1}{5}$.

If the denominator of the fraction contain a factor, which is a perfect square, it will be sufficient to multiply both terms by the other factor of the denominator. Thus, let it be required to find the square root of $\frac{7}{36}$; multiplying both terms by 4, the fraction becomes $\frac{28}{144}$, the root of which is $\frac{5}{12}$, accurate to within $\frac{1}{12}$.

If a greater degree of accuracy is required, we convert the fraction into another, the denominator of which is a perfect square, but greater than that obtained by the method above.

To find, for example, the square root of $\frac{3}{5}$ to within $\frac{1}{15}$, the fraction must be converted into 225ths. This is done by multiplying both terms by 45. Thus we have $\frac{3}{5}=\frac{135}{225}$, the root of which is nearest $\frac{12}{15}$, accurate to within $\frac{1}{15}$.

After making the denominator a perfect square, we may multiply both terms of the proposed fraction by any number, which is a perfect square, and thus approximate the root more nearly. If, for example, we multiply both terms of $\frac{15}{25}$ by 144, the square of 12, we obtain $\frac{2160}{3600}$, the root of which is nearest $\frac{46}{60}$. Thus, we have the root of $\frac{3}{5}$ to within $\frac{1}{60}$.

115. We may in this way approximate the roots of whole numbers, the roots of which cannot be exactly assigned.

11*

If it be required, for example, to find the square root of 2; we convert it into a fraction the denominator of which will be a perfect square. Thus, if we put $2=\frac{450}{225}$, we have for the root $\frac{21}{15}$ or $1\frac{6}{15}$, accurate to within $\frac{1}{15}$.

In general, to find the square root of a number accurate to within a given fraction, *we multiply the proposed number by the square of the denominator of the given fraction; we then find the entire part of the square root of this product, and divide the result by the denominator of the given fraction.*

This rule may be demonstrated as follows. Let a be the number proposed, and let it be required to find the root of a to within $\frac{1}{n}$.

We shall have it is evident, $a=\frac{a\,n^2}{n^2}$; let r be the entire part of the root of the numerator $a\,n^2$; $a\,n^2$ will be comprised between r^2 and $(r+1)^2$, and by consequence the square root of a will be comprised between those of $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$, that is to say, between $\frac{r}{n}$ and $\frac{r+1}{n}$; whence $\frac{r}{n}$ will be the root of a to within $\frac{1}{n}$.

116. To approximate the root of a number, which is not a perfect square, it will be most convenient to employ some power of 10 as the multiplier of the proposed, or which is the same thing, to convert the proposed into a fraction, the denominator of which shall be some power of 10. Thus, to approximate the root of 2, let us put $2=\frac{200}{100}$ or 2.00, the approximate root will be 1. 5. Again, let $2=\frac{20000}{10000}$ or 2.0000, the approximate root will be 1. 51.

117. From what has been done, and indeed from the nature of multiplication it follows, that the number of decimal places in the power will be double the decimal places in the

root. To find the approximate root of an entire number by the aid of decimals therefore, we must annex to this number twice as many zeros as there are decimal places wanted in the root. Thus, if 5 places are required in the root, ten zeros must be annexed. The zeros may be annexed as we proceed, it being observed, that two zeros must be annexed for every new figure placed in the root

The root of 7, to three places, will be found as follows

```
7 (2. 645
4
—
300
276
———
 2400
 2096
 ————
  30400
  26425
  —————
   3975
```

If the proposed be already a decimal, the number of decimal places must be made even by annexing a zero, if necessary. If the root of the number, thus prepared, is not sufficiently exact, two zeros must be annexed for every new figure required in the root.

118. To find the root of a vulgar fraction by the aid of decimals, we convert this fraction into a decimal and then extract the root.

If the proposed consist of an entire part and a fraction, we convert the fraction into a decimal, annex it to the entire part, and then extract the root.

In converting the fraction into a decimal, it will be necessary to pursue the operation, until twice as many decimals are obtained, as are wanted in the root.

EXAMPLES.

1. To find the root of $\frac{144}{624}$

2. To find the root of $13\frac{129}{400}$

3. To find the approximate root of $\frac{7}{9}$

4. To find the approximate root of $\frac{18}{28}$

5. To find the approximate root of $\frac{23}{38}$

6. To find the approximate root of 29

7. To find the approximate root of 4. 425

8. To find the approximate root of $\frac{11}{14}$

9. To find the approximate root of 0. 01001

10. To find the approximate root of $2\frac{13}{14}$

EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC QUANTITIES.

119. By the rule for multiplication, we have

$$(5\,a^2\,b^3\,c)^2 = 5\,a^2\,b^3\,c \times 5\,a^2\,b^3\,c = 25\,a^4\,b^6\,c^2$$

A simple quantity is therefore raised to the square by squaring the coefficient and doubling the exponent of each of the letters. Whence to extract the square root of a simple quantity, it is necessary 1°. *to extract the root of the coefficient.* 2°. *to divide the exponent of each of the letters by 2.*

According to this rule, we have

$$\sqrt{64\,a^6\,b^4} = 8\,a^3\,b^2$$
$$\sqrt{625\,a^2\,b^8\,c^6} = 25\,a\,b^4\,c^3$$

In order that a simple quantity may be a perfect square, its coefficient, it is evident from the preceding rule, must be a perfect square and the exponent of each of the letters must be an even number.

Thus $98\,a\,b^4$ is not a perfect square. Its root can therefore only be indicated by means of the radical sign, thus $\sqrt{98\,a\,b^4}$. Expressions of this kind are called *irrational* quantities of the second degree, or more simply *radicals* of the second degree.

120. The second power of a product, it is easy to see, is the same as the product of the second powers of all its factors. It follows therefore, *that the square root of a product will be the same as the product of the square root of all its factors.*

By means of this principle, we may frequently reduce to a more simple form expressions of the kind, which we are here considering. Thus, the above expression $\sqrt{98\,a\,b^4}$ may be put under the form $\sqrt{49\,b^4} \times \sqrt{2\,a}$; but $\sqrt{49\,b^4} = 7\,b^2$, whence $\sqrt{98\,a\,b^4} = 7\,b^2\sqrt{2\,a}$

In like manner, we have

$$\sqrt{864\,a^2\,b^5\,c^{11}} = \sqrt{144\,a^2\,b^4\,c^{10} \times 6\,b\,c} = 12\,a\,b^2\,c^5\sqrt{6\,b\,c}$$

In the expressions $7\,b^2\sqrt{2\,a}$, $12\,a\,b^2\,c^5\sqrt{6\,b\,c}$, the quantities $7\,b^2$, $12\,a\,b^2\,c^5$ placed without the radical sign are called the *coefficients* of the radical. The expressions themselves are said to be reduced to their most simple form.

From what has been done, we have the following rule for reducing irrational quantities, consisting of one term, to their most simple form, viz. *Separate the quantity proposed into two parts, one of which shall contain all the factors, which are perfect squares and the other those, which are not; write the roots of the factors, which are perfect squares without the radical sign as multipliers of the radical quantity, and retain under the radical sign the factors, which are not perfect squares.*

121. The square of $-a$, it will be observed, is a^2, as well as that of $+a$; the root therefore of a^2 may be either $+a$ or $-a$. Both of these roots may be comprehended in one expression by means of the double sign $\pm$. Thus $\sqrt{a^2} = \pm a$, $\sqrt{25\,b^4\,c^2} = \pm 5\,b^2\,c$.

The double sign, it is evident, should be considered as affecting the square root of all quantities whatever.

If the simple quantity proposed be negative, the square root is impossible; since there is no quantity positive or negative, which multiplied by itself will produce a negative quantity. Thus, $\sqrt{-a}$, $\sqrt{-3\,b^2}$ are *impossible* or *imaginary* quantities.

Expressions of this kind may be simplified in the same manner as radical expressions, which are real. Thus $\sqrt{-9}$ may be put under the form $\sqrt{-1 \times 9}$; whence

$$\sqrt{-9} = 3\sqrt{-1}.$$

In like manner $\sqrt{-4a^2} = 2a\sqrt{-1}$.

122. We proceed to the extraction of the square root of polynomials.

A quantity consisting of two terms cannot, it is evident, be a perfect square, for the square of a simple quantity will be a simple quantity, and the square of a binomial consists always of three terms.

This being premised, let the proposed be a trinomial, its root, it is evident, will consist of at least two terms. Let $m + n$ be the root, we have $(m+n)^2 = m^2 + 2mn + n^2$.

This shows, that if the proposed be arranged with reference to the powers of some letter that, 1°. the first term of the proposed will be the square of the first term of the root sought; 2°. the second term of the proposed will be equal to twice the first term of the root multiplied by the second; 3°. the third term of the proposed will be the square of the second term of the root.

Let it be proposed to extract the root of the trinomial

$$24a^2b^3c + 16a^4c^2 + 9b^6$$

Arranging with reference to the letter a, the proposed becomes

$$16a^4c^2 + 24a^2b^3c + 9b^6.$$

In order to obtain the root, we extract according to what has been said, the root of the first term $16a^4c^2$, which gives $4a^2c$. This is the first term of the root. Dividing next the second term $24a^2b^3c$ by $8a^2c$, twice the term of the root already found, we have $3b^2$ for the second term of the root, and since the square of this is equal to $9b^4$ the remaining term of the proposed, the proposed is a perfect square, the root of which is $4a^2c + 3b^2$.

Again, let the proposed consist of more than three terms, its root will consist of more than two terms. Let it consist

of three and let $m+n+p$ be the root. The expression $m+n+p$ may be put under the form $(m+n)+p$, forming the square after the manner of a binomial, we have for the result $(m+n)^2+2(m+n)p+p^2$, or, developing $(m+n)^2$, the result will be $m^2+2mn+n^2+2(m+n)p+p^2$. The proposed therefore being arranged with reference to the powers of some letter, it is evident, that the first term of the root will be found by extracting the root of the first term of the proposed, and that the second term of the root will be found by dividing the second term of the proposed by twice the first term of the root already found. If then we subtract from the proposed the square of the two terms of the root already obtained, the remainder will be equal to twice the first two terms of the root multiplied by the third plus the square of the third. Dividing this remainder therefore by twice the terms of the root already found, or which is the same thing, dividing the first term of the remainder by twice the first term of the root, we shall obtain the third term sought. Subtracting from the first remainder twice the product of the first two terms of the root by the third, together with the square of the third, if the result be 0, the proposed is a perfect square, and the root is exactly obtained.

Let it be proposed to find the square root of the polynomial $49a^2b^2-24ab^3+25a^4-30a^3b+16b^4$.

The proposed being arranged with reference to the letter a, the work will be as follows

$$\begin{array}{l|l}
25a^4-30a^3b+49a^2b^2-24ab^3+16b^4 & 5a^2-3ab+4b^2 \\
\underline{25a^4-30a^3b+\;\;9a^2b^2} & \overline{10a^2} \\
\qquad\qquad\qquad 40a^2b^2-24ab^3+16b^4 & \\
\qquad\qquad\qquad \underline{40a^2b^2-24ab^3+16b^4} & \\
\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}$$

We begin by extracting the root of $25a^4$, this gives $5a^2$ for the first term of the root sought, which we place at the right of the proposed and on the same line with it; we then multiply this term of the root by 2 and write the result $10a^2$

under the root. Dividing next the second term of the proposed by $10\,a^2$ we obtain $-3\,a\,b$ for the second term of the root sought. Squaring the part of the root already found, viz. $5\,a^2-3\,a\,b$, and subtracting the square from the proposed, we have for the first term of the remainder $40\,a^2\,b^2$. Dividing this last by $10\,a^2$ the double of $5\,a^2$, we obtain $4\,b^2$ for the quotient.

This is the third term of the root sought; forming next the double product of $5\,a^2-3\,a\,b$ by $4\,b^2$, and subtracting the result together with the square of $4\,b^2$ from the first remainder the result is 0. The proposed is therefore a perfect square, and we have for the root required

$$5\,a^2-3\,a\,b+4\,b^2.$$

The calculations in the above example may be performed with more facility as follows.

$$\begin{array}{l|l}
25\,a^4-30\,a^3\,b+49\,a^2\,b^2-24\,a\,b^3+16\,b^4 & 5\,a^2-3\,a\,b+4\,b^2 \\
\cline{2-2}
25\,a^4 & 10\,a^2-3\,a\,b \\
\cline{1-1}
\qquad -30\,a^3\,b+49\,a^2\,b^2 & 10\,a^2-6\,ab+4\,b^2 \\
\qquad -30\,a^3\,b+9\,a^2\,b^2 & \\
\cline{1-1}
\qquad\qquad\qquad 40\,a^2\,b^2-24\,a\,b^3+16\,b^4 & \\
\qquad\qquad\qquad 40\,a^2\,b^2-24\,a\,b^3+16\,b^4 & \\
\cline{1-1}
\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}$$

Having found the first term $5\,a^2$ of the root, we subtract its square from the first term of the proposed, and bring down the next two terms for a dividend. Dividing the first term of the dividend by $10\,a^2$, we obtain $-3\,a\,b$, the second term of the root; this we place by the side of $10\,a^2$; we then multiply the whole viz. $10\,a^2-3\,a\,b$ by this second term and subtract the result from the dividend, which gives a remainder $40\,a^2\,b^2$; to this remainder we bring down the two remaining terms of the proposed for a new dividend. Doubling the two terms of the root already found for a new divisor, we write the result under $10\,a^2$; dividing next the first term of the new dividend by the first term of the divisor, we ob-

tain $4\,b^2$ the third term of the root, which we place by the side of the last divisor ; we then multiply the whole by this last term of the root, aud subtracting the result from the last dividend 0 remains.

123. The same process it is easy to see, may be extended to a polynomial of any number of terms whatever.

EXAMPLES.

1. To find the square root of

$4\,a^4 + 12\,a^3\,x + 13\,a^2\,x^2 + 6\,a\,x^3 + x^4$

2. To find the square root of

$x^6 + 4\,x^5 + 10\,x^4 + 20\,x^3 + 25\,x^2 + 24\,x + 16$

3. To find the square root of

$25\,a^4\,b^2 - 40\,a^3\,b^2\,c + 76\,a^2\,b^2\,c^2 - 48\,a\,b^2\,c^3$
$+ 36\,b^2\,c^4 - 30\,a^4\,b\,c + 24\,a^3\,b\,c^2 - 36\,a^2\,b\,c^3 + 9\,a^4\,c^2$

4. To find the square root of

$$16\,a^4 - 40\,a^3\,b + 25\,b^2 \left|\, a^2 - 80\,a\,b^2\,c + 64\,b^2 c^2 \right.$$
$$64\,b\,c$$

124. The polynomial proposed being arranged with reference to the powers of some letter, if the first term of the proposed is not a perfect square, or if in the course of the operation we arrive at a remainder, the first term of which is not divisible by twice the first term of the root, the proposed is not a perfect square, and the root cannot be exactly assigned.

The polynomial $a^3\,b + 4\,a^2\,b^2 + 4\,a\,b^3$, for example, is not, it is easy to see, a perfect square ; the root therefore can only be indicated, thus, $\sqrt{a^3\,b + 4\,a^2\,b^2 + 4\,a\,b^3}$. We may, however, apply to expressions of this kind the same simplifications, that have already been applied to simple quantities. The proposed indeed may be put under the form $\sqrt{(a^2 + 4\,a\,b + 4\,b^2)\,a\,b}$; but the root of $a^2 + 4\,a\,b + 4\,b^2$ is evidently $a + 2\,b$, whence

$$\sqrt{a^3\,b + 4\,a^2\,b^2 + 4\,a\,b^3} = (a + 2\,b)\sqrt{a\,b}$$

EQUATIONS OF THE SECOND DEGREE.

125. An equation is said to be of the second degree, when it contains the second power of the unknown quantity without any of the higher powers.

In an equation of the second degree there can be therefore three kinds of terms only, viz. 1°. Terms, which involve the second power of the unknown quantity, 2°. terms, which involve the first power of the unknown quantity, 3°. terms consisting entirely of known quantities.

An equation, which contains all three of these different kinds of terms is called a *complete* equation of the second degree.

If the second of these different kinds of terms be wanting, the equation is then called an *incomplete* equation of the second degree.

A complete equation of the second degree is sometimes called an *affected* equation, and an incomplete equation is sometimes called a *pure* equation of the second degree.

126. We are now prepared for the solution of incomplete equations of the second degree.

Let there be proposed, for example, the equation

$$3x^2 - 29 = \frac{x^2}{4} + 510$$

Freeing from denominators, we have

$$12x^2 - 116 = x^2 + 2040$$

transposing and uniting terms

$$11x^2 = 2156$$

or

$$x^2 = 196$$

whence, extracting the root of both members

$$x = 14$$

Equations of the second degree, it is to be observed, admit of two values for the unknown quantity, while those of the first degree admit of but one only. This arises from the circumstance, that the second power of a quantity will be positive, whether the quantity itself be positive or negative.

Thus we have x in the preceding example equal $+14$ or -14, or, uniting both values in one expression, we have

$$x = \pm 14.$$

Let us take, as a second example, the equation

$$\frac{5}{7}x^2 - 8 = 4 - \frac{2}{3}x^2$$

Freeing from denominators, transposing and reducing, we have

$$x^2 = \frac{252}{29}, \quad \text{whence } x = \sqrt{\frac{252}{29}}$$

In this example $\frac{252}{29}$ is not a perfect square; we can therefore obtain only an approximate value for x.

Let us take, as a third example, the equation

$$x^2 + 25 = 9$$

Deducing the value of x from this equation, we have

$$x = \sqrt{-16}$$

To find the value of x, we are here required to extract the square root of -16. But this is impossible; for, as there is no quantity positive or negative, which multiplied by itself will produce a negative quantity, -16, it is evident, cannot have a square root either *exact* or *approximate*. -16 may indeed be considered as arising from the multiplication of $+4$ by -4; but $+4$ and -4 are different quantities; their product therefore is not a square.

The result $x = \sqrt{-16}$ shows then, that it is impossible to resolve the equation, which leads to this result. In general, an expression for the square root of a negative quantity is to be regarded as a symbol of *impossibility*.

127. Equations of the kind, which we are here considering, may always be reduced to an equation of the form $ax^2 = b$, a and b denoting any known quantities whatever, positive or negative. It is evident, that they may be reduced to this state, *by collecting into one member the terms, which involve* x^2 *and reducing them to one term, and collecting the known terms into the other member.*

Resolving the equation $a x^2 = b$, we have

$$x = \sqrt{\frac{b}{a}}$$

This is a general solution for incomplete equations of the second degree.

If $\frac{b}{a}$ be a perfect square, the value of x may be obtained exactly, if not, it may be found with such degree of approximation as we please. If $\frac{b}{a}$ be negative, we shall have $\sqrt{-\frac{b}{a}}$, a symbol of impossibility.

From what has been done, we have the following rule for the solution of incomplete equations of the second degree, viz. *Collect into one member all the terms, which involve the square of the unknown quantity and the known quantities into the other; free the square of the unknown quantity from the quantities, by which it is multiplied or divided; the value of the unknown quantity will then be obtained by extracting the square root of each member.*

QUESTIONS PRODUCING INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

1. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128.

2. It is required to divide the number 18 into two such parts, that the squares of these parts may be in the proportion of 25 to 16.

3. It is required to divide the number 14 into two such parts, that the quotient of the greater part divided by the less, may be to the quotient of the less divided by the greater as 16 to 9.

4. Two persons A and B lay out some money on speculation. A disposes of his bargain for £11 and gains as much per cent. as B lays out; B's gain is £36, and it appears that

A gains 4 times as much per cent. as B. Required the capital of each.

5. A charitable person distributed a certain sum among some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one third as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now the men received all together 18s. more than the women. How many were there of each?

6. In a court there are two square grass plots; a side of one of which is 10 yards longer than the side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

7. A person bought two pieces of linen, which together measured 36 yards. Each of them cost as many shillings a yard as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces?

8. There is a rectangular field, whose length is to the breadth in the proportion of 6 to 5. A part of this equal to $\frac{1}{6}$ of the whole being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

9. Two workmen A and B were engaged to work for a certain number of days at different rates. At the end of the time, A, who had played 4 of the days, received 75 shillings; but B, who had played 7 of the days received only 48 shillings. Now had B played 4 days, and A played 7 days, they would have received exactly alike. For how many days were they engaged; how many did each work, and what had each per day?

10. Two travellers A and B set out to meet each other, A leaving the town C at the same time that B left D. They travelled the direct road CD, and on meeting, it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What was the distance between C and D?

11. What two numbers are those, whose difference being multiplied by the greater, and the product divided by the less, the quotient is 24; but if their difference be multiplied by the less, and the product divided by the greater, the quotient is 6.

12. A and B carried 100 eggs between them to market and each received the same sum. If A had carried as many as B he would have received 18 pence for them, and if B had taken only as many as A, he would have received only 8 pence. How many had each?

13. What two numbers are those, whose difference multiplied by the greater produces 40, and multiplied by the less produces 15?

COMPLETE EQUATIONS OF THE SECOND DEGREE.

128. Let us take next the equation $x^2 + 8x = 209$. This is a complete equation of the second degree. The solution of this equation, it is evident, would present no difficulty, if the left hand member were a perfect square. But this is not the case; for, the square of a quantity consisting of one term will consist of one term, and the square of a quantity consisting of two terms will contain three terms. Let us then see if $x^2 + 8x$ can be made a perfect square; for this purpose, it will be recollected, that the three parts, which compose the square of a binomial are 1°. *the square of the first term of the binomial*, 2°. *twice the first term multiplied by the second*, 3°. *the square of the second term.* Thus

$$(x + a)^2 = x^2 + 2ax + a^2$$

If then we compare $x^2 + 8x$ with $x^2 + 2ax + a^2$, it is evident, that $x^2 + 8x$ may be considered the first and second terms in the square of a binomial. The first term of this binomial will evidently be x; then as $8x$ must contain twice the first term by the second, the second will be found by dividing $8x$ by $2x$, which gives 4 for the quotient. $x^2 + 8x$ is therefore the first two terms in the square of the binomial $x + 4$. If then we add 16 the square of 4 to $x^2 + 8x$, the

left hand member of the proposed, the result $x^2+8x+16$ will be a perfect square. But if 16 be added to the left hand member, it must also be added to the right, in order to preserve the equality; the proposed will then become

$$x^2+8x+16=225$$

Extracting the root of each member of this last, we have

$$x+4=\pm 15$$

whence $\qquad x=11, \quad x=-19$

Let us take, as a second example, the equation

$$x^2-\frac{2}{3}x=15\tfrac{8}{9}$$

Comparing $x^2-\frac{2}{3}x$ with the square of the binomial $x-a$, viz. $x^2-2ax+a^2$, it is evident, that $x^2-\frac{2}{3}x$ may be considered the first two terms of the square of a binomial. By the same course of reasoning as in the preceding example, we find this binomial to be $x-\frac{1}{3}$. If then the square of $\frac{1}{3}$ be added to both sides, the left hand member will be a perfect square, we have then

$$x^2-\frac{2}{3}x+\frac{1}{9}=16$$

Extracting the root of each member, we have

$$x-\frac{1}{3}=\pm 4$$

whence $\qquad x=4\tfrac{1}{3}, \quad x=-3\tfrac{2}{3}$

Let us take as a third example, the equation $x^2+px=q$.

Comparing the left hand member of this equation with $x^2+2ax+a^2$, it is evident, that it may be considered as the first two terms in the square of the binomial $x+\frac{p}{2}$; whence if the square of $\frac{p}{2}$ be added to both sides, the left hand member will become a perfect square, and we shall have

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4}$$

Extracting the root of each member

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}}$$

whence $\quad x=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}},\quad x=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}$

Making the left hand member a perfect square is called *completing the square.* This is done, as will readily be inferred from the preceding examples, *by adding to both sides the square of one half the coefficient of* x *in the second term.*

Let us take, for a fourth example, the equation

$$7-\frac{3}{5}x=\frac{61-x^2}{4x-2}$$

Freeing from denominators, we have

$$140x-70-12x^2+6x=305-5x^2$$

Transposing and uniting terms, we have

$$146x-7x^2=375$$

Or changing the signs of each term, and dividing by the coefficient of x^2

$$x^2-\frac{146x}{7}=-\frac{375}{7}$$

Completing the square, we have

$$x^2-\frac{146x}{7}+\frac{5329}{49}=-\frac{375}{7}+\frac{5329}{49}=\frac{2704}{49}$$

Whence extracting the root of each member

$$x-\frac{73}{7}=\pm\frac{52}{7}$$

$$x=17\tfrac{6}{7},\quad x=3$$

The rule for completing the square applies only, it is evident, to equations of the form $x^2+px=q$, p and q denoting any quantities whatever, positive or negative.

If not already of the form $x^2+px=q$, equations of the kind, which we are here considering, must always be reduced to this form, before completing the square. Thus, in the preceding example, the given equation was reduced, before

completing the square, to $x^2 - \frac{146}{7} x = - \frac{375}{7}$, an equation of the form required.

It is evident, that all complete equations of the second degree may be reduced to the form $x^2 + p x = q$, 1°. by collecting all the terms, which involve x into the first member and uniting the terms, which contain x^2 into one term, and those which contain x into another, 2°. by changing the signs of each term, if necessary, in order to render that of x^2 positive, 3°. by dividing all the terms by the multiplier of x^2, if it have a multiplier, and multiplying all the terms by the divisor of x^2, if it have a divisor.

Let the equation $\frac{a x}{4} - b x^2 = \frac{c x}{5} + a e$ be reduced to the form $x^2 + p x = q$

Freeing from denominators, we have

$$5 a x - 20 b x^2 = 4 c x + 20 a e$$

By transposition $\quad -20 b x^2 + 5 a x - 4 c x = 20 a e$

Changing signs $\quad 20 b x^2 - 5 a x + 4 c x = -20 a e$

Uniting terms $\quad 20 b x^2 - (5 a - 4 c) x = -20 a e$

Dividing by $20 b$ $\quad x^2 - \frac{(5 a - 4 c)}{20 b} x = - \frac{a e}{b}$

Comparing this equation with the general formula, we have $p = - \frac{(5 a - 4 c)}{20 b}$, $q = - \frac{a e}{b}$.

From what has been done, we have the following rule for the solution of complete equations of the second degree, viz. 1°. *The equation being reduced to the form* $x^2 + p x = q$, *add to both members the square of half the coefficient of* x *in the second term* ; 2°. *extract the square root of both members, taking care to give to the root of the second member the double sign* $\pm$; 3°. *deduce the value of* x *from the equation, which arises from the last operation.*

EXAMPLES.

1. Given $\frac{2 x^2}{3} + 3\frac{1}{2} = \frac{x}{2} + 8$, to find the values of x.

2. Given $4x - \frac{36 - x}{x} = 46$, to find the values of x.

3. Given $\frac{x}{x + 60} = \frac{7}{3x - 5}$, to find the values of x.

4. Given $\frac{x + 3}{2} + \frac{16 - 2x}{2x - 5} = 5\frac{1}{3}$, to find the values of x.

5. Given $\frac{4x - 5}{x} - \frac{3x - 7}{3x + 7} = \frac{9x + 23}{13x}$, to find the values of x.

6. Given $x + 4y = 14$ and $y^2 + 4x = 2y + 11$ $\Big\}$ to find the values of x and y.

7. Given $2x + 3y = 118$ and $5x^2 - 7y^2 = 4333$ $\Big\}$ to find the values of x and y.

8. Given $\frac{2x + 7}{4x} = 2y - \frac{51 + 2x}{10}$ and $\frac{4x + 3y}{16} = y - 2$ $\Bigg\}$ To find the values of x and y.

129. We pass next to the solution of some questions.

1. To find a number such, that if three times this number be added to twice its square, the sum will be 65.

Putting x for the number sought, we have by the question

$$2x^2 + 3x = 65$$

Dividing by 2, we have $x^2 + \frac{3}{2}x = \frac{65}{2}$

Completing the square $\quad x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{65}{2} + \frac{9}{16}$

Extracting the root $\quad x + \frac{3}{4} = \pm\frac{23}{4}$

whence $\quad x = 5, \quad x = -\frac{13}{2}$

The first value of x satisfies the question in the sense, in which it is enunciated. In order to interpret the second, it will be observed, that if we put $-x$ instead of x in the equation $2x^2 + 3x = 65$, it becomes $2x^2 - 3x = 65$. Resolving this equation, we obtain $x = \frac{13}{2}, x = -5$, values of x, which differ from the preceding only in the signs. The

number $\frac{13}{2}$ will therefore satisfy the conditions of the question modified, thus

To find a number such, that if three times this number be subtracted from twice its square, the remainder will be 65.

The negative value here modifies the proposed question, in a manner analogous to what takes place, as we have already seen, in equations of the first degree.

2. To find a number such, that if 15 be added to its square the sum will be equal to eight times this number.

Putting x for the number sought, we have by the question $$x^2+15=8x$$

Resolving this equation, we have

$$x=5, \quad x=3$$

In this example both values of x are positive, and answer directly the conditions of the question, in the sense in which it is enunciated.

3. To find a number such, that if the square of this number be augmented by 5 times the number and also by 6, the result will be 2.

Putting x for the number sought, we have by the question $$x^2+5x+6=2.$$

Whence, resolving the equation we have

$$x=-1, \quad x=-4$$

The values of x in this example are both negative; the question therefore, as is evident from inspection, cannot be solved in the sense, in which it is enunciated.

If instead of x we write $-x$ in the equation of the proposed it becomes $x^2-5x+6=2$, from which we obtain $x=1$, $x=4$. The numbers 1 and 4 will therefore satisfy the conditions of the proposed modified thus,

To find a number such, that if five times this number be subtracted from its square, and 6 be added to the remainder, the result will be 2.

4. To divide the number 10 into two such parts, that the product of these parts will be 30.

Putting x for one of the parts $10-x$ will be the other; we have therefore by the question

$$10x-x^2=30$$

Resolving this equation, we obtain

$$x=5+\sqrt{-5}. \quad x=5-\sqrt{-5}$$

This result indicates, that there is some absurdity in the conditions of the question proposed, since in order to obtain the value of x, we must extract the root of a negative quantity, which is impossible.

In order to see in what this absurdity consists, let us examine into what two parts a given number should be divided, in order that the product of these parts may be the greatest possible.

Let us represent the given number by p, the product of the two parts by q, and the difference of the two parts by d; the greater part will then be $\frac{p}{2}+\frac{d}{2}$ and the less $\frac{p}{2}-\frac{d}{2}$, and we shall have

$$\left(\frac{p}{2}+\frac{d}{2}\right)\left(\frac{p}{2}-\frac{d}{2}\right)=q$$

or

$$\frac{p^2}{4}-\frac{d^2}{4}=q$$

By diminishing the value of d in this last, we increase, it is evident, the value of q; the value of q will therefore be the greatest possible when d is zero, that is, *the product will be the greatest possible, when the difference between the two parts is zero, or in other words, when the two parts are equal.*

The greatest possible product, which can be obtained by dividing 10 into two parts and taking their product will be 25. The absurdity of the question above consists therefore in requiring, that the product of the two parts, into which 10 is to be divided, should be greater than 25.

130. The following questions will serve as an exercise for the learner.

1. A merchant sold a quantity of brandy for $39, and gained as much per cent. as the brandy cost him. What was the price of the brandy?

2. There are two numbers, whose difference is 9, and their sum multiplied by the greater produces 266. What are those numbers?

3. Two partners A and B gained £18 by trade. A's money was in trade 12 months, and he received for his principal and gain £26. Also B's money, which was £30, was in trade 16 months. What money did A put into trade?

4. A person bought some sheep for £72; and found if he had bought 6 more for the same money, he would have paid £1 less for each. How many did he buy, and what was the price of each?

5. A and B set out from two towns, which were distant 247 miles, and travelled the direct road till they met. A went 9 miles a day; and the number of days at the end of which they met, was greater by 3, than the number of miles, which B went in a day. How many miles did each go?

6. A person bought two pieces of cloth of different sorts; whereof the finer cost 4 shillings a yard more than the other; for the finer he paid £18; but the coarser, which exceeded the finer in length by 2 yards, cost only £16. How many yards were there in each piece, and what was the price of a yard of each?

7. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the same journey. How many miles did he go per hour?

8. Three merchants A, B, and C made a joint stock, by which they gained a sum less than that stock by £80; A's share of the gain was £60, and his contribution to the stock was £17 more than B's. Also B and C contributed together £325. How much did each contribute?

9. Two messengers A and B were dispatched at the same time to a place 90 miles distant; the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him. At what rate did each travel per hour?

10. The joint stock of two partners A and B was £416. A's money was in trade 9 months and B's six months; on dividing their stock and gain A received £228, and B £252. What was each man's stock?

11. What two numbers are those, whose sum is to the greater as 11 to 7; the difference of their squares being 132?

12. A and B hired a pasture, into which A put 4 horses and B as many as cost him 18 shillings a week. Afterwards B put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

13. A vintner sold 7 dozen of sherry and 12 dozen of claret for $50. He sold 3 dozen of sherry more for $10 than he did of claret for $6. Required the price of each?

14. What number is that, which being divided by the product of its two digits the quotient is 2, and if 27 be added to it, the digits will be inverted?

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND DEGREE.

131. All complete equations of the second degree may, as we have already seen, be reduced to an equation of the form $x^2+px=q$, p and q denoting any known quantities whatever positive or negative. Resolving this equation, we have

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}.$$

This is a general solution for equations of the second degree. We shall now examine the circumstances, which result from the different hypotheses, which may be made upon the known quantities p and q. This is the object *of the discussion of the general equation of the second degree.*

132. Before proceeding to this discussion, however, we shall first demonstrate, that every equation of the second degree admits of two values for the unknown quantity and of two only. In order to this, we take the general equation

$$x^2+px=q \quad (1)$$

completing the square, we have

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4}$$

or $$\left(x + \frac{p}{2}\right)^2 = q + \frac{p^2}{4}$$

Let $q + \frac{p^2}{4} = m^2$, we shall then have

$$\left(x + \frac{p}{2}\right)^2 = m^2$$

or $$\left(x + \frac{p}{2}\right)^2 - m^2 = 0$$

But the first member of this equation, being the difference between two squares, may be put under the form

$$\left(x + \frac{p}{2} + m\right)\left(x + \frac{p}{2} - m\right)$$

we have therefore

$$\left(x + \frac{p}{2} + m\right)\left(x + \frac{p}{2} - m\right) = 0 \quad (2)$$

The first member of this last being composed of two factors, if either of the factors be equal to 0, the whole member will become 0, and the equation will be satisfied.

If $x + \frac{p}{2} - m = 0$, then $x = -\frac{p}{2} + m$

If $x + \frac{p}{2} + m = 0$, then $x = -\frac{p}{2} - m$

Or substituting for m its value, we have

$$x = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}$$

$$x = -\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}$$

The first member of equation (2), it is evident can become zero only by putting for x a value, which will reduce to zero one of the two factors in which it is found. Since then equation (2) is a necessary consequence of equation (1) and the converse, it follows *that every equation of the sec-*

ond degree admits of two values for the unknown quantity and of two only.

Either of the two values for x, it will be observed, taken separately will satisfy the equation. They cannot, however, be introduced together, for, being different, their product cannot be x^2.

133. Let us now proceed to the discussion proposed. Resuming the value of x obtained from the general equation $x^2+px=q$, we have

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}$$

In order to find the value of this expression, which contains a radical, that is, a quantity the root of which is to be extracted, we must be able to extract the root either exactly or by approximation ; $q+\frac{p^2}{4}$, the quantity placed under the radical sign, must therefore be positive. But $\frac{p^2}{4}$ will necessarily be positive, whatever the sign of p may be ; the sign of the quantity $q+\frac{p^2}{4}$ will therefore depend principally upon that of q, or the quantity in the equation altogether known.

1. This being premised let q in the first place be *positive*. In this case the general equation will be of the form

$$x^2\pm px=+q$$

and we shall have

$$x=\mp\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}$$

Here $q+\frac{p^2}{4}$ will evidently be positive ; the value of x may therefore be obtained, either exactly or with such degree of approximation as we please.

With respect to the two values of x, the first, viz.

$$x=\mp\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}$$

will be *positive*, for, the square root of $\frac{p^2}{4}$ alone being $\frac{p}{2}$, the square root of $q+\frac{p^2}{4}$ will be greater than $\frac{p}{2}$, the value of x will therefore have the same sign with the radical and will by consequence be positive. This value will answer directly the conditions of the equation, or the problem of which the equation is the algebraic translation.

The second value of x, viz. $x=\mp\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}$, being also necessarily of the same sign with the radical, will be essentially *negative*. This value, though it satisfies the equation, will not answer the conditions of the question, from which the equation is derived. It belongs to an analogous question corresponding to the equation, after $-x$ has been introduced instead of x, that is, to $x^2\mp p\,x=q$. Indeed, from this last equation, we deduce $x=\pm\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}$, values, which do not differ from the preceding except in the sign. Thus the same equation connects together two questions, which differ from each other only in the sense of certain conditions.

2. Again, let q be *negative*. The equation will then be of the form $x^2\pm p\,x=-q$, and we have

$$x=\mp\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$$

Here in order that the root of the quantity placed under the radical sign may be taken, it is evidently necessary, that q should not exceed $\frac{p^2}{4}$.

Since moreover $\sqrt{\frac{p^2}{4}-q}$ is numerically less than $\frac{p}{2}$, it follows, that the values of x will both be *negative*, if p is positive in the equation, that is, if the equation is of the form $x^2+p\,x=-q$; and that they will both be *positive*, if p is negative in the equation, that is, if the equation is of the form $x^2-p\,x=-q$.

Indeed, it may be shown *a priori*, that always *when* q *is negative in the second member and* p *negative in the first, the problem will admit of two direct solutions, provided that* q *does not exceed* $\frac{p^2}{4}$.

The equation $x^2 - p x = -q$ may, by changing the signs of all the terms, be put under the form

$$p x - x^2 = q, \text{ or } x(p - x) = q$$

But the equation $x(p-x) = q$ is evidently the algebraic translation of the following enunciation, viz. *To divide a number* p *into two parts, the product of which shall be equal to a given number* q. For if we put x for one of the parts, the other part will be $p - x$, and the product of the two parts will be $x(p-x)$.

This being premised, the enunciation of the problem admits, it is evident, of two direct solutions; for the equation of the problem will be the same, whether x be put for one or the other of the parts; there is no reason then, why the equation, when resolved, should give one of the parts rather than the other; it should therefore give both at the same time.

Moreover, in order that the problem may be possible, it is necessary, that q should not exceed $\frac{p^2}{4}$; for the greatest possible product of the parts, into which the number p may be divided being equal only to $\frac{p^2}{4}$, it is absurd to require that their product, which we have represented by q, should be greater than $\frac{p^2}{4}$. We conclude therefore that, in all cases *when the known quantity is negative in the second member, but numerically greater than the square of half the coefficient of the second term, the question proposed is impossible.*

The questions art. 129 are particular examples of the cases, which we have here been considering.

EXAMINATION OF PARTICULAR CASES.

1. In the general equation let q be negative, that is, let the equation be of the form $x^2 + px = -q$, p being of any sign whatever; if we suppose $q = \frac{p^2}{4}$, the radical

$$\sqrt{\frac{p^2}{4} - q}$$

will be reduced to 0, and the values of x will be equal each to $-\frac{p}{2}$. Thus if q be negative in the equation and equal to $\frac{p^2}{4}$, the values of x will be equal, and will both be positive if p is negative, or both negative if p is positive.

2. In the general formula

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q},$$

let $q = 0$, the values of x will then be $x = 0$, $x = -p$.

3. In the same formula let $p = 0$, we have then

$$x = \pm \sqrt{q},$$

that is to say, the values of x will in this case be equal, but of contrary signs, real if q is positive, and imaginary if q is negative.

4. Let $p = 0$, $q = 0$, the values of x will then be each equal to 0.

5. We have next to examine a remarkable case, which frequently occurs in the solution of problems of the second degree. For this purpose, let us take the equation

$$ax^2 + bx = c.$$

This equation, being resolved, gives

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

Let it now be supposed, that in consequence of a particular hypothesis made upon the given things in the question, we have $a = 0$, the values of x then become

$$x = \frac{0}{0}, \quad x = -\frac{2b}{0}$$

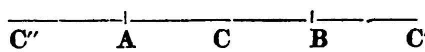
The second value of x here presents itself under the form of infinity and may be regarded as a true answer, when the question is susceptible of infinite solutions. In order to interpret the first, if we return to the equation, we see that the hypothesis $a=0$ reduces it to $bx=c$, from which we deduce $x=\frac{b}{c}$, an expression *finite* and *determinate*, and which must be regarded as the true value of $\frac{0}{0}$ in the present case.

6. Let it be supposed finally, that we have at the same time $a=0$, $b=0$, $c=0$, The equation will then be altogether indeterminate. This is the only case of indetermination, which the equation of the second degree presents.

DISCUSSION OF PROBLEMS.

134. The following problems offer all the circumstances, which usually occur in problems of the second degree.

1. To find on the line A B, which joins two luminous bodies A and B, the point, where these bodies shine with equal light.



The solution of this problem depends upon the following principle in physics, viz. The intensity of light from the same luminous body will be, at different distances, in the inverse ratio of the square of the distance.

This being premised, let $a=$ A B the distance between the two bodies ; let $b=$ the intensity of A at the unit of distance, $c=$ the intensity of B at the same distance ; let C be the point required, and let A C $=x$.

The intensity of A at the distance 1 being b, its intensity at the distance 2, 3, 4 . . . will be $\frac{b}{4}$, $\frac{b}{9}$, $\frac{b}{16}$ . . ., and by consequence, at the distance x, it will be $\frac{b}{x^2}$. For the

same reason, the intensity of B at the distance $a-x$ will be $\frac{c}{(a-x)^2}$; whence by the question, we have

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}$$

From which, we obtain

$$x=\frac{ab}{b-c}\pm\sqrt{\frac{a^2b^2}{(b-c)^2}-\frac{a^2b}{b-c}}$$

or reducing $$x=\frac{a(b\pm\sqrt{bc})}{b-c}$$

But $b\pm\sqrt{bc}$ may, it will be observed, be put under the form $\sqrt{b}(\sqrt{b}\pm\sqrt{c})$, and $b-c$ may be put under the form $(\sqrt{b})^2-(\sqrt{c})^2$ or $(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})$. Taking advantage of this remark, the value of x may be expressed more simply, thus

$$x=\frac{a\sqrt{b}}{\sqrt{b}\pm\sqrt{c}},\quad\text{whence}\quad a-x=\frac{\pm a\sqrt{c}}{\sqrt{b}\pm\sqrt{c}}$$

or
$$\left.\begin{aligned}x&=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}\\ x&=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}\end{aligned}\right\}\quad\text{and}\quad\left\{\begin{aligned}a-x&=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}\\ a-x&=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}\end{aligned}\right.$$

DISCUSSION.

1. Let b be greater than c.

The first value of x is positive and less than a since $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a fraction. The point sought therefore, according to this value of x, is situated between A and B. It is moreover nearer to B than to A; for, in consequence of $b>c$, we have $\sqrt{b}+\sqrt{b}$ or $2\sqrt{b}>\sqrt{b}+\sqrt{c}$, whence $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$, and by consequence $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$. This indeed should be the case, since we have supposed the intensity of A greater than that of B.

The corresponding value of $a-x$ is also positive and less, as it will be easy to see, than $\frac{a}{2}$.

The second value of x is positive, but greater than a, since we have $\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}} > 1$. This value of x gives therefore a second point C′ situated upon A B produced and at the right of A and B. Indeed, since the light from A and B expands itself in all directions, there should be, it is easy to see, on A B produced a second point where A and B shine with equal light. This point moreover should be nearer the body, the light of which is least intense.

The second value of $a-x$ is negative, this should be the case, since we have $x > a$.

2. Let b be less than c.

The first value of x is positive, but less than $\frac{a}{2}$. The corresponding value of $a-x$ is also positive and greater than $\frac{a}{2}$.

Thus in the present hypothesis the point C, situated between A and B should be nearer A than B.

The second value of x is essentially negative. In order to interpret it, we return to the equation, which becomes by substituting $-x$ for x, $\frac{b}{x^2}=\frac{c}{(a+x)^2}$. But $a-x$ expressing in the first instance the distance of the point sought from B, $a+x$ must in the present case express the same distance. Thus the point sought should be at the left of A, in C″ for example. Indeed, since by hypothesis the intensity of B is greater than that of A, the second point sought should be nearer A than B.

3. Let $b=c$.

The first value of x, and also that of $a-x$ is reduced in this case to $\frac{a}{2}$. Thus we have the middle of A B for the point sought. This result conforms to the hypothesis.

The remaining values are reduced to $\frac{a\sqrt{b}}{0}$, or become infinite, that is, the second point, where the bodies shine

with equal light, is situated at a distance from A and B greater than any assignable quantity. This result corresponds perfectly with the present hypothesis; for, if we suppose the difference $b - c$, instead of being absolutely nothing to be very small, the second point will exist, but at a very great distance from A and B. If then $b = c$ or

$$\sqrt{b} - \sqrt{c} = 0$$

the point required must cease to exist or be placed at an infinite distance.

4. Let $b = c$ and $a = 0$.

The first system of values of x and $a - x$ reduce themselves in this case to 0, and the second system to $\frac{0}{0}$. This last character is here the symbol of indetermination; for, on returning to the equation of the problem

$$(b - c) x^2 - 2abx = - a^2 b$$

this equation becomes on the present hypothesis

$$0 . x^2 - 0 . x = 0,$$

an equation, which may be satisfied by any number whatever taken for x. Indeed, since the two bodies have the same intensity and are placed at the same point, they should shine with equal light upon any point whatever in the line AB.

5. Finally let $a = 0$, b being different from c.

Both systems in this case will be reduced to 0, which indicates, that there is but one point, where the bodies shine with equal light, viz. the point, in which the two bodies are situated.

2. To find two numbers such, that the difference of their products by the numbers a and b respectively may be equal to a given number s, and the difference of their squares equal to another given number q.

Denoting by x and y the numbers sought, we have by the question

$$a x - b y = s$$
$$x^2 - y^2 = q$$

Resolving these equations, we have for the first system of values for x and y.

$$x=\frac{as+b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$$

$$y=\frac{bs+a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$$

and for the second system, we have

$$x=\frac{as-b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$$

$$y=\frac{bs-a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$$

DISCUSSION

1. Let a be greater than b, and by consequence a^2-b^2 positive.

In order that the values of x and y may be real, it is necessary that we have

$$q(a^2-b^2)<s^2, \text{ and therefore } q<\frac{s^2}{a^2-b^2}$$

This condition being fulfilled, the values of x and y in the first system will be necessarily positive, and will by consequence form a *direct solution* of the problem in the sense, in which it is enunciated.

In the second system the value of x will be essentially positive; for, $a>b$ gives $as>bs$ and for a still stronger reason $as>b\sqrt{s^2-q(a^2-b^2)}$.

With respect to the value of y, it may be either positive or negative. In order that it may be positive, we must have

$$bs>a\sqrt{s^2-q(a^2-b^2)}$$

or, squaring both sides

$$b^2s^2>a^2s^2-a^2q(a^2-b^2)$$

or, adding $a^2q(a^2-b^2)$ to both sides of this last, and subtracting b^2s^2 from both sides

$$a^2q(a^2-b^2)>s^2(a^2-b^2)$$

or, by division $$q > \frac{s^2}{a^2}$$

Thus in order that the second system may be a *real and direct solution*, we must have

$$q < \frac{s^2}{a^2 - b^2}, \text{ but } q > \frac{s^2}{a^2}.$$

If then we take for a, b, and s any absolute numbers whatever provided that $a > b$, and that we take for q a number comprised between the two limits $\frac{s^2}{a^2}$ and $\frac{s^2}{a^2 - b^2}$, we shall be certain of obtaining *two direct solutions.*

Thus, let $a = 6$, $b = 4$, $s = 15$; we have

$$\frac{s^2}{a^2} = \frac{225}{36} = 6\tfrac{1}{4}, \text{ and } \frac{s^2}{a^2 - b^2} = \frac{225}{20} = 11\tfrac{1}{4};$$

if then we take $q = 10$, for example, we shall have

$$x = \frac{6 \times 15 \pm 4\sqrt{225 - 20 \times 10}}{20} = \frac{11}{2} \text{ or } \frac{7}{2}$$

$$y = \frac{4 \times 15 \pm 6\sqrt{225 - 20 \times 10}}{20} = \frac{9}{2} \text{ or } \frac{3}{2}$$

If on the present hypothesis, we have $q < \frac{s^2}{a^2}$, and for a still stronger reason $q < \frac{s^2}{a^2 - b^2}$, the value of y in the second system will be negative. This system therefore will not be a solution of the proposed problem in the sense, in which it is enunciated, but of an analogous problem, the equations of which are

$$a x + b y = s$$
$$x^2 - y^2 = q$$

and which will differ from the proposed in this respect only, that s will express an arithmetical *sum* instead of *difference.*

2. Let a be less than b and therefore $a^2 - b^2$ *negative.*

In this case the expressions for x and y in the first system may be put under the form

$$x=\frac{-as-b\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$$

$$y=\frac{-bs-a\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$$

and in the second

$$x=\frac{-as+b\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$$

$$y=\frac{-bs+a\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$$

The values of x and y in both systems, it is evident, will be real, since the quantity placed under the radical is essentially positive.

In the first system the values of x and y are essentially *negative*; in the second the value of x, it is easy to see, is necessarily positive, but the value of y may be either positive or negative; in order that it may be positive, we must have $q>\frac{s^2}{a^2}$.

3. Let $a=b$, and therefore $a^2-b^2=0$.

On this hypothesis, we have for the first system of values for x and y

$$x=\frac{2as}{0},\ y=\frac{2as}{0}$$

and for the second

$$x=\frac{0}{0},\ y=\frac{0}{0}$$

Returning to the equations of the proposed in order to interpret these last, we obtain for x and y on the present hypothesis

$$x=\frac{a^2q+s^2}{2as},\ y=\frac{a^2q-s^2}{2as}$$

QUESTIONS FOR SOLUTION AND DISCUSSION.

1. There are two numbers, whose sum is a and the sum of whose second powers is b. Required the numbers.

2. To find two numbers such, that their difference may be a, and that the square of the greater may be equal to the product of the less multiplied by a given number p.

3. To find two numbers such, that the sum of their products by the numbers a and b respectively may be equal to $2s$, and their product equal to p.

4. To find a number such, that the square of this number may be to the product of the differences between this number and two other numbers a and b in the ratio of m to n.

MAXIMA AND MINIMA.

135. In several of the preceding questions, the given things, we have seen, are so connected among themselves, that one is determined by the others to be comprised within certain limits, or to have a greatest or least possible value.

A quantity, the value of which may be made to vary, is called a *variable* quantity; the greatest value of which is called a *maximum* and the least a *minimum*.

Questions frequently occur, in which it is required to determine under what circumstances the result of certain arithmetical operations performed upon numbers will be the greatest or least possible. We shall resolve a few questions of this kind, the solutions of which depend upon equations of the second degree.

1. To divide a number $2a$ into two parts such, that the product of these parts may be a *maximum*.

Let x be one of the parts, then $2a - x$ will be the other, and their product will be $x(2a - x)$. By assigning different values to x, the product $x(2a - x)$ will vary in magnitude, and the question is to assign to x a value such, that this product may be the greatest possible. Let m be the maximum sought, we have by the question

$$x(2a - x) = m.$$

Regarding for the moment m as known, and deducing from this equation the value of x, we have

$$x=a\pm\sqrt{a^2-m}$$

From this result it appears, that in order that x may be real, m must not exceed a^2; the greatest value of m will therefore be a^2, in which case we have $x=a$. Thus *to obtain the greatest possible product, the proposed must be divided into two equal parts, and the maximum obtained will be equal to the square of one of these parts.*

In the equation $x(2a-x)=m$, the expression $x(2a-x)$ is called a *function* of x. This function is itself a *variable*, the value of which depends upon that given to the first variable or x.

2. To divide a number $2a$ into two parts such, that the sum of the square roots of these parts may be a *maximum*.

Let x^2 be one of the parts, then $2a-x^2$ will be the other, and the sum of the square roots will be $x+\sqrt{2a-x^2}$. Let m be the maximum sought, we have by the question

$$x+\sqrt{2a-x^2}=m$$

from which we obtain

$$x=\frac{m}{2}\pm\sqrt{\frac{m^2}{4}+\frac{2a-m^2}{2}}$$

or simplifying $$x=\frac{m}{2}\pm\frac{1}{2}\sqrt{4a-m^2}$$

In order that the values of x may be real, the value of m^2 must not exceed $4a$; $2\sqrt{a}$ is therefore the greatest value, which m can receive.

Let us put $m=2\sqrt{a}$, we have $x=\sqrt{a}$ and $x^2=a$, whence $2a-x^2=a$. Thus, *the proposed must be divided into two equal parts in order that the sum of the square roots of the parts should be a maximum.* This maximum moreover will be equal to *twice the square root of one of the parts.*

3. Let it be proposed next to find for x in the expression $\frac{p^2x^2+q^2}{(p^2-q^2)x}$ a value such as to render this expression a *minimum*.

Putting $\frac{p^2x^2+q^2}{(p^2-q^2)x}=m$, we have

$$p^2x^2-(p^2-q^2)\,m\,x=-q^2$$

from which we obtain

$$x=\frac{(p^2-q^2)\,m}{2p^2}\pm\frac{1}{2p^2}\sqrt{(p^2-q^2)^2m^2-4p^2q^2}$$

In order that the values of x may be real $(p^2-q^2)^2m^2$ must at least be equal to $4p^2q^2$, and by consequence m must at least be equal to $\frac{2pq}{p^2-q^2}$. Putting $m=\frac{2pq}{p^2-q^2}$ in the expression for x, the radical disappears, and we have

$$x=\frac{p^2-q^2}{2p^2}\times\frac{2pq}{p^2-q^2}=\frac{q}{p}$$

The least value of the proposed is therefore $\frac{2pq}{p^2-q^2}$, and the value of x, which will render the proposed a minimum, is $x=\frac{q}{p}$.

From what has been done, the following rule for the solution of questions of the kind, which we are here considering will readily be inferred, viz. *Having formed the algebraic expression of the quantity susceptible of becoming a* maximum *or* minimum, *make this expression equal to any quantity whatever* m. *If the equation thus obtained is of the second degree in* x, x *designating the variable quantity, which enters into the algebraic expression, resolve this equation in relation to* x ; *make next the quantity under the radical equal to zero, and deduce from this last equation the value of* m ; *this will be the* maximum *or* minimum *sought. Substituting finally the value of* m *in the expression for* x, *we obtain the value of* x *proper to satisfy the enunciation proposed.*

If the quantity placed under the radical remains essentially positive, whatever the value of m, we infer that the expression proposed may be of any assignable magnitude whatever, or in other words, that it will have *infinity for a maximum* and *zero for a minimum.*

Thus let there be proposed the expression $\frac{4x^2+4x-3}{6(2x+1)}$; to determine whether this expression is susceptible of a *maximum* or *minimum*.

Putting $\frac{4x^2+4x-3}{6(2x+1)}=m$ and deducing the value of x, we have $x=\frac{3m-1}{2}\pm\frac{1}{2}\sqrt{9m^2+4}$. Here, whatever value we give to m, the quantity placed under the radical will be positive; the proposed therefore may be of any magnitude whatever.

EXAMPLES FOR PRACTICE.

1. To divide a given number a into two factors, the sum of which shall be a *minimum*.

2. Let d be the difference between two numbers; required that the square of the greater divided by the less may be a *minimum*.

Ans. The minimum required is $4d$, and the value of x corresponding is $2d$.

3. Let a and b be two numbers of which a is the greater, to find a number such that if a be added to this number and b be subtracted from it, the product of the sum and difference thus obtained being divided by the square of the number; the quotient will be a *maximum*.

Ans. The number $=\frac{2ab}{a-b}$, and the maximum $=\frac{(a+b)^2}{4ab}$

4. To divide a number $2a$ into two parts such, that the quotients obtained by dividing the parts mutually one by the other may be a *minimum*.

Ans. The number should be divided into two equal parts, and the minimum is 2.

5. To find a number such that if a and b be added to this number respectively, the product of the two sums thus obtained divided by the number may be a *minimum*.

Ans. The number $=\sqrt{ab}$, and the minimum.

$$=(\sqrt{a}+\sqrt{b})^2$$

6. To find the least value of the expression

$$\frac{1}{a+x}+\frac{1}{a-x}$$

7. To find the least value of the expression

$$(a+x)+\frac{(a-x)^2}{a+x}.$$

Section V.

OF THE FORMATION OF POWERS AND THE EXTRACTION OF THEIR ROOTS.

136. When a quantity is multiplied into itself the product, we have seen, is called a *power*, the degree of which is marked by the exponent of the product, thus $aaaaa$ or a^5 is called the fifth power of a; in like manner a^m is called the mth power of a.

The original quantity, from which a power is derived, is called the root of this power. The degree of the root is determined by the number of times the root is found as a factor in the power; thus a is the fifth root of a^5; in like manner a is the mth root of a^m. The number, which marks the degree of the root is called the *index* of the root.

POWERS AND ROOTS OF SIMPLE QUANTITIES.

137. Let it be proposed to find the fifth power of $2\,a^3\,b^2$; this power is indicated thus, $(2\,a^3\,b^2)^5$, and we have, it is evident,

$$(2\,a^3\,b^2)^5=2\,a^3\,b^2\times 2\,a^3\,b^2\times 2\,a^3\,b^2\times 2\,a^3\,b^2\times 2\,a^3\,b^2$$

Here, it is evident, 1°. that the coefficient 2 must be multiplied into itself four times or raised to the fifth power, 2°. that each one of the exponents of the letters must be added, until it is taken as many times as there are units in the exponent of the power, or in other words multiplied by 5; we have therefore

$$(2\,a^3\,b^2)^5=32\,a^{15}\,b^{10}$$

In like manner $(8\,a^2\,b^3\,c)^3=512\,a^6\,b^9\,c^3$

To raise a simple quantity therefore to any given power, *we raise the coefficient to this power, and multiply each one of the exponents of the letters by the exponent of the power.*

With respect to the sign, with which the powers of a simple quantity should be affected, it is evident, that whatever be the sign of the quantity itself, its second power will be *positive.* Moreover if the exponent of the power of a simple quantity be an even number, it is easy to see, that this power may be considered as a power of the square of the proposed quantity. Thus a^8, it is evident, may be considered the fourth power of a^2; in like manner a^{2m}, any even power of a, may be considered the mth power of a^2. It follows therefore, *that whatever be the sign of a simple quantity, any power of this quantity, the exponent of which is an even number, is positive.*

Again, since the power of a simple quantity, the exponent of which is an odd number is equal to a power of this quantity of an even degree multiplied by the first power, it follows, *that every power of a simple quantity, the exponent of which is an odd number, will have the same sign as the quantity from which it is formed.*

138. Let it now be proposed to find the third root of $64\,a^6\,b^9\,c^3$. The root required is indicated, thus, $\sqrt[3]{64\,a^6\,b^9\,c^3}$; the sign $\sqrt{}$ being employed to denote in general that a root is to be taken, and the index 3 placed above the radical sign to denote the particular root required.

Since the root of a quantity must evidently be sought by a process the reverse of that, by which it is raised to a power, in order to extract the root of a simple quantity, 1°. *we extract the root of the coefficient.* 2°. *we divide the exponent of each of the letters by the index of the root.*

According to this rule the third root of the proposed will be $4\,a^2\,b^3\,c$. In like manner the seventh root of $a^{14}\,b^{21}\,c^{56}$ is $a^2\,b^3\,c^8$.

With respect to the signs, with which the roots of simple quantities should be affected, it is an evident consequence of the principles already established that,

1°. *Every root of an even degree of a simple positive quantity*

may have indifferently either the sign + or —. Thus the sixth root of $64\,a^{18}$ is $\pm 2\,a^3$.

2°. *Every root, the degree of which is expressed by an odd number, will have the same sign as the quantity proposed.* Thus the fifth root of $-32\,a^{10}\,b^5$ is $-2\,a^2\,b$.

3°. *Every root of an even degree of a negative simple quantity is an impossible or imaginary root.* For there is no quantity, which raised to a power of an even degree can give a negative result.

Thus $\sqrt[4]{-a}$, $\sqrt[6]{-b}$ denote impossible or imaginary quantities, in the same manner as $\sqrt{-a}$, $\sqrt{-b}$.

139. From what has been said, it is evident in order that a root may be extracted, 1°. that the coefficient of the proposed must be a perfect power of the degree marked by the index of the root to be extracted. 2°. that the exponents of each of the letters must be divisible by the index of the root.

When this is not the case the root can only be indicated. It is to be observed, however, that radical expressions, of any degree whatever admit of the same simplifications as those of the second degree. These simplifications are founded upon the principle, *that any root whatever of a product is equal to the product of the same root of the several factors.*

Thus let it be proposed to find the third root of $54\,a^4\,b^3\,c^2$. The third root, it is evident, cannot be taken; for 54 is not a perfect third power, and the exponents of the letters a and c are not divisible by 3. We therefore indicate the root, thus, $\sqrt[3]{54\,a^4\,b^3\,c^2}$; but this expression may be put under the form $\sqrt[3]{27\,a^3\,b^3 \times 2\,a\,c^2}$; whence extracting the third root of the factor $27\,a^3\,b^3$, we have

$$\sqrt[3]{54\,a^4\,b^3\,c^2} = 3\,a\,b\,\sqrt[3]{2\,a\,c^2}$$

In like manner, we have $\sqrt[5]{96\,a^5\,b^7\,c^{11}} = 2\,a\,b\,c^2\,\sqrt[5]{3\,b^2\,c}$.

POWERS AND ROOTS OF COMPOUND QUANTITIES.

Powers of compound quantities are found like those of simple quantities by the continued multiplication of the quan-

o itself. They are indicated by inclosing the quan- a parenthesis, to which is annexed the exponent of the power. The third power of $a^2 + 5ab - bc^2$, for example, is indicated, thus, $(a^2 + 5ab - bc^2)^3$. This same power may also be indicated, thus, $\overline{a^2 + 5ab - bc^2}|^3$

Next to simple quantities binomials are those, which are the least complicated. We begin therefore with these.

Below are several of the first powers of the binomial $x + a$, viz.

$$(x+a)^1 = x + a$$
$$(x+a)^2 = x^2 + 2ax + a^2$$
$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$
$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$

We have formed the different powers of $x + a$ in this table by the continued multiplication of $x + a$ into itself. In this way we arrive only at particular results. To form any of the higher powers, the process of multiplication must still be continued. This would be tedious especially as the power, to which the binomial is to be raised, becomes more and more elevated. We proceed therefore to investigate a method, by which a binomial may be raised to any power whatever without the necessity of forming the inferior powers. This method was discovered by Newton. The principle on which it is founded is called the *Binomial Theorem*. The most simple and elementary demonstration of this theorem depends upon the theory of combinations, to which we shall first attend.

THEORY OF COMBINATIONS.

140. The results, obtained by writing one after the other in every possible way a given number of letters, in such a manner, that all the letters will enter into each result, are called *permutations*.

Let there be, for example, two letters a and b. These give, it is evident, two permutations ab, ba.

Again, let there be three letters a, b, and c. If we set

apart one of the letters a, for example, the remaining letters give two permutations, viz. $b\,c$, $c\,b$; placing next the a at the right of each of these, we have two permutations of three letters, viz. $b\,c\,a$, $c\,b\,a$; but each of the remaining letters b and c, being set apart in the same manner, will also furnish each two permutations of three letters; *whence the permutations of three letters will be equal to the permutations of two letters multiplied by three.*

In like manner *the permutations of four letters will be found equal to the permutations of three letters multiplied by four.*

We infer therefore that, in general, *the permutations of any number whatever* n *of letters will be equal to the permutations of* n — 1 *letters, multiplied by* n *the number of letters employed.*

Let Q represent the permutations of $n-1$ letters, then $Q\,n$ will represent the permutations of n letters; thus $Q\,n$ will be a general formula for permutations.

In the general formula Qn let $n=2$, then Q will be 1; whence 1×2 will be the permutations of two letters. Again let $n=3$, then Q will be 1×2; whence $1\times 2\times 3$ will be the permutations of three letters. In like manner the permutations of 4 letters will be $1\times 2\times 3\times 4$. The following rule for permutations will therefore be readily inferred, viz. *Multiply in order the natural numbers* 1, 2, 3, 4, &c. *to the number denoting the letters employed inclusive*; *the result will be the permutations of the given number of letters.*

141. When a given number of letters are disposed in order one after the other in every possible way, 2 and 2, 3 and 3, and, in general, n and n at a time, the number of letters taken at a time being always less than the given number of letters, the results obtained are called *arrangements.*

Let it be required to form the arrangements of three letters, a, b, and c, two and two at a time.

Setting apart one of the letters a, for example, we write after this letter each one of the reserved letters b and c, we thus form two of the arrangements required, viz. $a\,b$, $a\,c$; but each of the letters b and c, being set apart in the same manner, will also furnish with the letters reserved two of the

arrangements required; we thus obtain all the arrangements of three letters taken 2 and 2; *whence the arrangements of three letters 2 and 2 at a time will be equal to the arrangements of the same letters one at a time, multiplied by the number of letters reserved.*

Again, let there be four letters a, b, c, d, to be arranged three and three at a time.

Forming, as above, the arrangements of the given letters, 2 and 2 at a time, the number of reserved letters will be 2. If then we take one of these arrangements $a\,b$, for example, and write successively each one of the reserved letters c, d by its side, we shall form two of the arrangements required, viz. $a\,b\,c$, $a\,b\,d$. The same being done with each one of the arrangements of the given letters taken 2 and 2 at a time, we shall obtain the whole number of arrangements of the same letters taken 3 and 3 at a time; *whence the arrangements of 4 letters taken 3 and 3 at a time will be equal to the arrangements of the same letters taken 2 and 2 at a time multiplied by the number of letters reserved.* We infer therefore that, in general, *the arrangements of any number* m *of letters, taken* n *and* n *at a time, will be equal to the arrangements of the same number of letters,* n — 1 *at a time, multiplied by the number of letters reserved.*

Let P represent the arrangements of m letters $n-1$ at a time; the reserved letters will then be $m-(n-1)$ or $m-n+1$; the arrangements of m letters, n and n at a time, will then be expressed by the formula $P\,(m-n+1)$. This will therefore be the general formula for arrangements.

In the general formula $P\,(m-n+1)$, let n equal 2. In this case P will represent the arrangements of m letters 1 at a time; thus P will equal m; whence $m\,(m-1)$ will express the arrangements of m letters 2 and 2 at a time.

Again, in the general formula $P\,(m-n+1)$, let $n=3$. In this case P will represent the arrangements of m letters 2 and 2 at a time; P will therefore equal $m\,(m-1)$; whence $m\,(m-1)\,(m-2)$ will express the arrangements of m letters 3 and 3 at a time.

In like manner the arrangements of m letters 4 and 4 at a time will be expressed by $m(m-1)(m-2)(m-3)$.

From inspection of the above formulas the following rule for arrangements will be readily inferred, viz. *From the number denoting the given letters subtract successively the natural numbers* 1, 2, 3, &c. *to the number, which denotes the letters to be taken at a time; multiply these several remainders and the number denoting the given letters together; the product will be the arrangements required.*

142. Arrangements, any two of which differ at least by one of the letters, which enter into them, are called *combinations.*

Let it be proposed to determine the number of combinations of three letters a, b, and c taken two and two at a time. The arrangements of these letters, two and two at a time, will be

$$\begin{array}{c} a\,b \\ b\,a \\ \hline a\,c \\ c\,a \\ \hline b\,c \\ c\,b \end{array}$$

Among these arrangements we have, it is evident, but three combinations, viz. $a\,b$, $a\,c$, $b\,c$, each one of which is repeated as many times, as there are permutations of two letters. Hence *the combinations of three letters taken* 2 *and* 2 *at a time, will be equal to the arrangements of three letters* 2 *and* 2 *at a time, divided by the permutations of two letters.*

In a similar manner it may be shown, that the combinations of 4 letters, 3 and 3 at a time, will be equal to the arrangements of 4 letters, 3 and 3 at a time, divided by the permutations of three letters.

We infer therefore that, in general, *the combinations of* m *letters,* n *and* n *at a time, will be equal to the arrangements of* m

letters, n *and* n *at a time, divided by the permutations of* n *letters*.

From what has been done, we have therefore the following general formula for combinations, viz.

$$\frac{P(m-n+1)}{Qn}$$

In the general formula let $n=2$, the formula, which results, will be $\frac{m(m-1)}{1.\ 2}$.

This will give the combinations of m letters 2 and 2 at a time.

Again, let $n=3$; the formula, which results will be

$$\frac{m(m-1)(m-2)}{1.\ 2.\ 3.}.$$

This will give the combinations of m letters 3 and 3 at a time.

In like manner we obtain $\frac{m(m-1)(m-2)(m-3)}{1.\ 2.\ 3.\ 4}$, a formula which gives the combinations of m letters 4 and 4 at a time.

From inspection of the formulas obtained by making $n=2, 3, 4,$ &c. in the general expression, we may infer a general rule for combinations, as has been done already with respect to permutations and arrangements.

EXAMPLES.

1. For how many days can 7 persons be placed in a different position at dinner?

2. How many words can be made with 5 letters of the alphabet, it being admitted that a number of consonants may make a word?

3. How many combinations can be made of 24 letters of the alphabet, taken two and two at a time?

4. A general was asked by his king, what reward he should confer on him for his services; the general only desired a farthing for every file of 10 men in a file, which he

could make with a body of 100 men. At this rate, would he receive ?

BINOMIAL THEOREM.

143. If we examine with attention the different powers of $x+a$ art. 139, it will be easy to fix upon the law, according to which the exponents of x and a proceed. But it will not be so easy to determine the law for the numerical coefficients. If we observe, however, the manner in which the different terms, which compose a power are formed, we shall perceive that the numerical coefficients are occasioned by the reduction of several similar terms into one, and that these similar terms arise from the equality of the factors, which compose a power. These reductions, it is easy to see, will not take place, if the second terms of the binomials are different. We begin therefore by investigating a law for the formation of the product of any number of binomials $x+a$, $x+b$, $x+c$ . . . , the first terms of which are the same in each, while the second are different.

$$(x+a)(x+b)=x^2+\left.\begin{array}{c}a\\b\end{array}\right|x+ab$$

$$(x+a)(x+b)(x+c)=x^3+\left.\begin{array}{c}a\\b\\c\end{array}\right|x^2+\left.\begin{array}{c}ab\\ac\\bc\end{array}\right|x+abc$$

$$(x+a)(x+b)(x+c)(x+d)=x^4+\left.\begin{array}{c}a\\b\\c\\d\end{array}\right|x^3+\left.\begin{array}{c}ab\\ac\\ad\\bc\\bd\\cd\end{array}\right|x^2+\left.\begin{array}{c}abc\\abd\\acd\\bcd\end{array}\right|+abcd$$

From inspection of the above products, which we have formed by the common rules of multiplication, it will be observed,

1°. *In each product there is one term more than there are units in the number of factors.*

2°. *The exponent of* x *in the first term is the same as the number of factors, and goes on decreasing by unity in each of the following terms.*

3°. *The coefficient of the first term is unity. The coefficient of the second term is equal to the sum of the second terms of the binomials; that of the third term is equal to the sum of the different combinations or products of the second terms of the binomials taken two and two; that of the fourth is equal to the sum of the products of the second terms of the binomials taken three and three and so on. The last term is equal to the product of the second terms of the binomials.*

144. We readily infer from analogy, that the same law will obtain, whatever be the number of factors employed. This law may however readily be shown to be general. In order to this, it will be sufficient to show, that if the law be true for the product of any number m of binomials, it will also be true for the product of $m+1$ binomials.

The number of binomial factors being represented by m, the different powers of x will be x^m, x^{m-1}, x^{m-2}, &c. Let A, B, C . . . U denote the quantities, by which these powers beginning with x^{m-1} are to be multiplied; but as the number of terms must remain indeterminate, until m receives a particular value, we can write only the first and last terms of the expression, designating the intermediate terms by a series of points.

The product of any number m of factors will then be represented by the expression.

$$x^m + \mathrm{A}x^{m-1} + \mathrm{B}x^{m-2} + \mathrm{C}x^{m-3} \ldots . \mathrm{U}$$

Multiplying this expression by a new factor $x+\mathrm{K}$, it becomes

$$x^{m+1} + \begin{array}{c}\mathrm{A}\\ \mathrm{K}\end{array}\left| x^m + \begin{array}{c}\mathrm{B}\\ \mathrm{AK}\end{array}\right| x^{m-1} + \begin{array}{c}\mathrm{C}\\ \mathrm{BK}\end{array}\left| x^{m-2} \ldots . \mathrm{UK}\right.$$

Here the law for the exponents is evidently the same, as in the first expression. With respect to the coefficients, it is evident, 1°. that the coefficient of the first term is unity, 2°. $\mathrm{A}+\mathrm{K}$, the coefficient of the second term, is equal to the sum of the second terms of the $m+1$ binomials. 3°. since B by hypothesis expresses the sum of the second terms of the m binomials taken two and two, and AK expresses the

sum of the second terms of the m binomials multiplied each by the new second term K, B + A K, the coefficient of the third term, will be the sum of the products two and two of the second terms of the $m+1$ binomials.

In the same manner C + B K, it is easy to see, will be the sum of the products three and three of the second terms of the $m+1$ binomials, and so on. 4°. The last term U K, it is evident, is the product of the $m+1$ second terms.

The law laid down art. 143 being true therefore for expressions of the fourth degree will, from what has just been demonstrated, be true for those of the fifth; and being true for expressions of the fifth degree, it will be true for those of the sixth and so on; thus it is general.

145. If in the different products, which have been formed art. 143, we make b, c and d each equal to a, these products will be converted into powers of $x+a$, thus

$$(x+a)(x+b)=(x+a)^2=x^2+\left.\begin{array}{r}a\\a\end{array}\right|x+a^2$$

$$(x+a)(x+b)(x+c)=(x+a)^3=x^3+\left.\begin{array}{r}a\\a\\a\end{array}\right|x^2+\left.\begin{array}{r}a^2\\a^2\\a^2\end{array}\right|x^2+a^3$$

$$(x+a)(x+b)(x+c)(x+d)=(x+a)^4$$

$$=x^4+\left.\begin{array}{r}a\\a\\a\\a\end{array}\right|x^3+\left.\begin{array}{r}a^2\\a^2\\a^2\\a^2\\a^2\\a^2\end{array}\right|x^2+\left.\begin{array}{r}a^3\\a^3\\a^3\\a^3\end{array}\right|x+a^4$$

Comparing these expressions with the different products, from which they are derived, we perceive 1°. that the multiplier of x in the second term has been converted into the first power of a, repeated as many times as there are units in the number of binomials employed, or which is the same thing, as there are units in the exponent of x in the first term. 2°. that the multiplier of the third term has been converted into the second power of a, repeated as many times, as there can be formed different products from a number of

letters, equal to the number of binomials employed, taken two and two at a time. 3°. that the multiplier of the fourth term has been converted into the third power of a, repeated as many times as there can be formed different products from a number of letters, equal to the number of binomials employed, taken three and three at a time, and so on.

146. From what has been done it is evident therefore, that whatever be the power to which a binomial $x + a$ is to be raised, 1°. the exponent of x in the first term will be equal to the exponent of the power, and that it will go on decreasing by unity in each of the following terms to the last, in which it will be 0. 2°. that the exponent of a in the first term will be 0, in the second unity, and that it will go on increasing by unity, until it becomes equal to the exponent of the power to be formed. 3°. the numerical coefficient of x in the first term will be unity, in the second it will be equal to the exponent of x in the first term, in the third it will be equal to the number of products, which may be formed from a number of letters, equal to the exponent of x in the first term, taken two and two at a time, in the fourth it will be equal to the number of products, which may be formed from the same number of letters, taken three and three at time and so on

Let it be required to form the 5th power of $x + a$. The different terms, without the numerical coefficients, will be by the preceding rule.

$$x^5 + a x^4 + a^2 x^3 + a^3 x^2 + a^4 x + a^5$$

With respect to the numerical coefficients, that of the first term will be 1, that of the second will be 5, that of the third will be equal to the number of products, which may be formed of 5 letters taken 2 and 2, that of the fourth will be equal to the number of products, which may be formed of 5 letters taken 3 and 3, and so on. Thus the numerical coefficients will be

$$1, \quad 5, \quad 10, \quad 10, \quad 5, \quad 1$$

whence

$$(x + a)^5 = x^5 + 5 a x^4 + 10 a^2 x^3 + 10 a^3 x^2 + 5 a^4 x + a^5$$

147. If it now be required to raise $x+a$ to the mth power, we shall have, according to the preceding rule, for a few of the first terms without the numerical coefficients

$$x^m + a\,x^{m-1} + a^2\,x^{m-2} + a^3\,x^{m-3} + \ldots$$

Here the numerical coefficients cannot be determined until we assign a particular value to x; by the preceding rule, however, the numerical coefficient of the second term will be equal to m, whatever the value of m may be. In the development therefore of $(x+a)^m$ we write m for the coefficient of the second term. With respect to the third term the numerical coefficient will be equal to the number of products, which may be formed of m letters 2 and 2 at a time; this is expressed by the formula $\frac{m\,(m-1)}{1.2}$, we write therefore $\frac{m\,(m-1)}{1.2}$ for the coefficient of the third term. For a similar reason $\frac{m\,(m-1)\,(m-2)}{1.2.3}$ will be the coefficient of the fourth term and so on. We have then

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1.2}\,a^2x^{m-2} + \frac{m(m-1)(m-2)}{1.2.3}\,a^3x^{m-3} \ldots\ldots a^m.$$

From inspection of the different terms of this development, it will be perceived, that the coefficient of the fourth, for example, is formed by multiplying $\frac{m\,(m-1)}{1.2}$, the coefficient of the third term, by $m-2$ the exponent of x in this term, and dividing by 3 the number, which marks the place of this term. It will be perceived also that the coefficient of the third term is formed in the same manner by means of the second term, and that of the second by means of the first. We readily infer therefore the following rule, by which to form the coefficient of any term whatever, viz. *Multiply the coefficient of the preceding term by the exponent of* x *in this term, and divide the product by the number, which marks the place of this term.*

From what has been done, we have therefore the follow-

ing rule, by which to raise a binomial to any power whatever, viz. 1°. *The coefficient of* x *in the first term is unity, and its exponent is equal to the number of units in the degree of the power to which the binomial is to be raised.* 2°. *to pass from one term to the following, we multiply the numerical coefficient by the exponent of* x *in the first, divide by the number which marks the place of this term, increase by unity the exponent of* a *and diminish by unity the exponent of* x.

According to this rule

$$(x+a)^6 = x^6 + 6\,ax^5 + 15\,a^2x^4 + 20\,a^3x^3 + 15\,a^4x^2 + 6\,a^5x + a^6$$

148. It sometimes happens, that the terms of the proposed binomial are affected with coefficients and exponents. The following example will exhibit the course to be pursued in cases of this kind.

Let it be proposed to raise the binomial $4\,a^2b - 3\,a\,b\,c$ to the fourth power.

Putting $4\,a^2b = x$, and $-3\,a\,b\,c = y$, we have

$$(x+y)^4 = x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,x\,y^3 + y^4$$

Substituting next for x and y their values, we have

$$(4\,a^2b - 3\,a\,b\,c)^4 = (4\,a^2b)^4 + 4\,(4\,a^2b)^3(-3\,a\,b\,c) + \dots$$
$$6\,(4\,a^2b)^2(-3\,a\,b\,c)^2 + 4\,(4\,a^2b)(-3\,a\,b\,c)^3 + (-3\,a\,b\,c)^4$$

or performing the operations indicated, we have

$$(4\,a^2b - 3\,a\,b\,c)^4 = 256\,a^8b^4 - 768\,a^7b^4c + 864\,a^6b^4c^2 \dots$$
$$- 432\,a^5b^4c^3 + 81\,a^4b^4c^4$$

The terms produced by this development are alternately positive and negative. This, it is evident, should always be the case, when the second term of the proposed binomial has the sign —.

149. The powers of any polynomial whatever may be found by the binomial theorem. Let it be proposed to find, for example, the third power of the trinomial $a + b + c$.

In order to apply the rule to this case, we put $a + b = m$; the proposed is then reduced to the binomial $m + c$, and we have $$(m+c)^3 = m^3 + 3\,m^2c + 3\,m\,c^2 + c^3$$

whence, restoring the value of m, we have

$$\begin{aligned}(a+b+c)^3 = a^3 &+ 3a^2b + 3ab^2 + b^3 \\ &+ 3a^2c + 6abc + 3b^2c \\ &+ c^3\end{aligned}$$

The same process, it is easy to see, may be applied to any polynomial whatever.

MISCELLANEOUS EXAMPLES.

1. To find the third power of $2a - b + c^2$.
2. To find the seventh power of $3a^5 - 2a^2$.
3. To find the fifth power of $a^2 - c - 2d$.
4. To find the fifth power of $7a^2b^5 - 10a^4c^2$.
5. To find the third power of $2a^2 - 4ab + 3b^2$.

150. We pass next to the extraction of the roots of compound quantities, beginning with the third or cube root of numbers.

In the following table, we have the nine first numbers, with their third powers or cubes written under them respectively.

1,	2,	3,	4,	5,	6,	7,	8,	9
1,	8,	27,	64,	125,	216,	343,	512,	729

By inspection of this table, it will be perceived, that among numbers consisting of two or three figures, there are nine only, which are perfect cubes, the others have each for a root an entire number plus a fraction.

If the proposed number consists of not more than three figures, its cube root, or that of the greatest cube contained in it, may be found immediately by the above table.

Let it be proposed to extract the cube root of a number, consisting of more than three figures 103823, for example.

The proposed being comprised between 1000, the cube of 10, and 1000000, the cube of 100, its root will consist of two places units and tens. To return therefore from the proposed to its root, let us observe the manner, in which the units and tens of a number are employed in forming the cube of

this number. For this purpose designating the tens by a and the units by b, we have

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

From this we learn, that the cube of a number consisting of units and tens is composed of *the cube of the tens, the triple product of the square of the tens by the units, the triple product of the tens by the square of the units, and the cube of the units.*

If then we can determine in the proposed the cube of the tens, the tens of the root will be found by extracting the cube root of this part. The cube of the tens, it is evident, can have no significant figure below the fourth place, the three figures on the right will therefore form no part of the cube of the tens, and may on this account be separated from the rest by a comma. The cube of the tens will then be contained in 103, the part at the left of the comma. The greatest cube contained in 103 is 64, the root of which is 4; 4 is therefore the significant figure in the tens of the root sought. Indeed, the proposed is evidently comprised between 64000, the cube of 40 or 4 tens, and 125000 the cube of 50 or 5 tens. The root sought is therefore composed of 4 tens and a certain number of units less than ten.

The tens of the root being thus obtained, we subtract the cube 64 from 103, the part of the proposed at the left of the comma, and to the remainder bring down the figures at the right. The result of this operation, 39823, must contain, from what has been said, *the triple product of the square of the tens by the units* together with the two remaining parts in the cube of the root sought.

The square of the tens, it is evident, will contain no significant figure less than hundreds, on this account we separate 23, the two figures on the right of the remainder 39823, from the rest by a comma; 398, the figures on the left of the comma, will then contain the triple product of the square of the tens of the root sought by the units and something more, in consequence of the hundreds arising from the two remaining parts of the cube of the root sought. Dividing

therefore 398 by 48, the triple product of the square of the tens, already found, the quotient 8 will be the unit figure sought, or from what has been said it may be too large by 1 or 2.

To determine whether 8 be the right unit figure we raise 48 to the cube. This gives 110592, a number greater than the proposed; 8 is therefore too large for the unit figure. We next try 7; 47 raised to the cube gives 103823. The proposed is therefore a perfect cube, the root of which is 47.

The operation may be exhibited as follows.

103,823	47
64	
398,23	48
103,823	

Any number however large may be considered as composed of units and tens; the process for finding the root may therefore be reduced to that of the preceding example.

Let it be proposed, for example, to find the cube root of 43725678. Considering the root of this number as composed of units and tens, 678 the three left hand figures, it is evident, will form no part of the cube of the tens. On this account we separate them from the rest by a comma. The cube of the tens being contained then in the part at the left of the comma, we obtain the tens of the root sought by extracting the cube root of this part. Considering therefore, for the moment, the part of the proposed 43725 as a separate number, its cube root, it is evident, may be found as in the preceding example. Performing the operations, we have 35 for the root and a remainder 850. There will therefore be 35 tens in the root of the proposed, and in order to find the units, we bring down the three right hand figures 678 by the side of 850, which gives 850678. Separating next the two right hand figures of this last from the rest by a comma, and dividing the part on the left by the triple square of the

tens already found, we obtain 2 for the unit figure of the root sought. To determine whether this is the right figure, we raise 352 to the cube, which gives 43614208, a result less than the proposed. 352 is therefore the root of the proposed to within a unit.

The operation may be exhibited as follows

	43,725,678	352	
	27		
1st Dividend	67,25	27	1st Divisor
Cube of 35	42875		
2d Dividend	850670	36575,	2d Divisor
Cube of 352	43614208		
Remainder	11450		

The same process, it is easy to see, may be extended to any number however large. The following rule therefore for the extraction of the third root will be readily inferred, viz. 1°. *Separate the number into periods of three figures each beginning at the right. The left hand period may consist of one two or three figures.* 2°. *Find the greatest third power in the left hand period, and write the root in the place of a quotient. Subtract the power from the period. To the remainder bring down the first figure of the next period for a dividend. Multiply the square of the root already found by three to form a divisor. See how many times the divisor is contained in the dividend, and write the result in the root. Raise the root thus augmented to the third power. If this is greater than the first two periods, diminish the quotient by one or more, until you obtain a third power, which may be subtracted from the first two periods. Perform the subtraction, and to the right of the remainder bring down the first figure of the next period to form a dividend and divide it by three times the second power of the two figures of the root, and write the quotient in the root. Then raise the whole root so found to the third power, and if it is not too large, subtract it from the first three periods; if it is too large, diminish the root as before. To the remainder*

bring down the first figure of the fourth period and perform the same series of operations as before.

If it happens, that the divisor is not contained in the dividend prepared as above, a zero must be placed in the root, and the next figure brought down to form the dividend.

EXAMPLES.

1. To find the third root of 91632508641.
2. To find the third root of 32977340218432.
3. To find the third root of 217125148004864.

151. If the proposed be a fraction its third root is found by extracting the third root of the numerator and denominator. Thus $\sqrt[3]{\frac{8}{27}}$ is $\frac{2}{3}$.

If the denominator is not a perfect third power it may be made so, by multiplying both terms by the square of the denominator; thus if the proposed be $\frac{3}{7}$, we multiply both terms by 49; the fraction then becomes $\frac{147}{343}$, the root of which is nearest $\frac{5}{7}$, accurate to within $\frac{1}{7}$.

152. We have seen, art. 111, that the square root of an entire number, which is not a perfect square, cannot be exactly assigned. The same is true with respect to the roots of all entire numbers, which are not perfect powers of a degree denoted by the index of the root.

The cube root of a number, which is not a perfect cube, may be approximated by converting the number into a fraction, the denominator of which is a perfect cube. Thus let it be required to find the approximate cube root of 15. This number may be put under the form $\frac{15 \times 12^3}{12^3} = \frac{25920}{1728}$, the cube root of which is $\frac{29}{12}$ or $2\frac{5}{12}$, accurate to within $\frac{1}{12}$. If a greater degree of accuracy were required, we should con-

vert the proposed into a fraction, the denominator of which is the cube of some number greater than 12.

In such cases it is most convenient to convert the proposed number into a fraction, the denominator of which shall be the cube of 10, 100, 1000, &c.

Thus if it be required to find the cube root of 25 to within .001, we convert the proposed into a decimal, the denominator of which is the cube of 1000, viz. 25.000000000, the third root of which is 2.920 to within .001; we have then $\sqrt[3]{25} = 2.920$ accurate to within .001.

To approximate therefore the cube root of an entire number by means of decimals, *We annex to the proposed three times as many zeros as there are decimal places required in the root, we then extract the root of the number thus prepared to within a unit, and point off for decimals, as many places as there are decimal figures required in the root.*

153. If the proposed number contain decimals, beginning at the place of units, we separate the number both to the right and left into periods of three figures each, annexing zeros if necessary to complete the right hand period in the decimal part. We then extract the root, and point off for decimals in the root as many places as there are periods in the decimal part of the power.

If the proposed be a vulgar fraction, the most simple method of finding the cube root is to convert the proposed into a decimal, the number of places in which shall be equal to three times the number of decimal figures required in the root. The question is thus reduced to extract the cube root of a decimal fraction.

EXAMPLES.

1. To find the approximate cube root of 79.
2. To find the approximate cube root of $15\frac{3}{4}$.
3. To find the approximate cube root of 3.00415.
4. To find the approximate cube root of 23.1762.
5. To find the approximate cube root of $\frac{14}{25}$.

154. By processes altogether similar to that, which we have employed in the extraction of the third root of numbers, we may extract the root of any degree whatever. The method of extracting the root of any degree whatever, in the case of algebraic quantities, is also founded upon the same principles. The following example will be sufficient to illustrate the course to be pursued, whatever the degree of the root required may be.

Let it be proposed to extract the fifth root of the polynomial

$$32\,a^{10} - 80\,a^8\,b^3 + 80\,a^6\,b^6 - 40\,a^4\,b^9 + 10\,a^2\,b^{12} - b^{15}$$

The proposed being arranged with reference to the powers of the letter a, we seek the fifth root of the first term $32\,a^{10}$. Its root $2\,a^2$ will be the first term of the root sought. We write therefore $2\,a^2$ in the place of the quotient in division, and subtracting its fifth power from the whole quantity, we have for a remainder

$$-80\,a^8\,b^3 + 80\,a^6\,b^6 + \&c.$$

The second term of the binomial $(a+b)^5$ is $5\,a^4\,b$; this shows, that in order to obtain the second term of the root, we must divide $-80\,a^8\,b^3$, the second term of the proposed, by five times the 4th power of $2\,a^2$, the term of the root already found. Performing the operation we obtain $-b^3$. This will be therefore the second term of the root. Raising $2\,a^2 - b^3$ to the fifth power, it produces the quantity proposed. The root is therefore obtained exactly. If the root contained more than two terms, it would be necessary to subtract the fifth power of $2\,a^2 - b^3$ from the proposed quantity, and then in order to find the next term of the root, to divide the first term of the remainder by five times the 4th power of $2\,a^2 - b^3$. In this case, however, only the first term of the divisor would be used; we should have therefore the same divisor, that was used the first time.

155. When the index of the root has divisors the root may be found more readily than by the general method. Thus the fourth root may be found by extracting the square

root twice successively; for the square root of a^4 is a^2, and that of a^2 is a, the fourth root of a^4. In general, all roots of a degree marked by 4, 8 or any power of 2 may be found by successive extractions of the square root. Roots, the indices of which are not prime numbers, may be reduced to others of a degree less elevated. The 6th root, for example, may be found by first extracting the square and then the cube root; for the square root of a^6 is a^3, and the cube root of a^3 is a.

EXAMPLES.

1. To find the cube root of

$$x^6 - 6x^5 - 40x^3 + 96x - 64$$

2. To find the cube root of

$$15x^4 - 6x + x^6 - 6x^5 - 20x^3 + 15x^2 + 1$$

3. To find the fourth root of

$$216a^2x^2 - 216ax^3 + 81x^4 + 16a^4 - 96a^3x$$

CALCULUS OF RADICAL EXPRESSIONS.

156. Radical expressions, the roots of which cannot be found exactly, frequently occur in the solution of questions. On this account mathematicians have been led to investigate rules for performing upon quantities subjected to the radical sign, the operations designed to be performed upon their roots. In this way the calculations required in the solution of a question are frequently rendered more simple, and the extraction of the root is left to be performed at last, when the radical expression is reduced to the most simple form, which the nature of the question will allow.

ADDITION AND SUBTRACTION.

157. Radical expressions of the same degree, and which have the quantities placed under the radical sign also the same, are said to be *similar*.

The addition and subtraction of similar radicals is per-

formed upon the coefficients. Thus the sum of the radicals $3\sqrt[3]{b}$, $9\sqrt[3]{b}$ is $12\sqrt[3]{b}$; the sum of $a\sqrt[5]{b^2c}$, $b\sqrt[5]{b^2c}$, $-c\sqrt[5]{b^2c}$ is $(a+b-c)\sqrt[5]{b^2c}$.

In like manner $9\sqrt[3]{a^4c}$ subtracted from $12\sqrt[3]{a^4c}$ gives $3\sqrt[3]{a^4c}$, and $b\sqrt[7]{ab^2}$ subtracted from $a\sqrt[7]{ab^2}$ gives

$$(a-b)\sqrt[7]{ab^2}$$

Radical expressions, which are at first dissimilar frequently become similar, when reduced to their most simple form. Thus, let it be required to add $5\sqrt[3]{2a^5b^2}$ and $a\sqrt[3]{54a^2b^2}$. These expressions reduced to their most simple form become $5a\sqrt[3]{2a^2b^2}$, $3a\sqrt[3]{2a^2b^2}$; their sum is therefore

$$8a\sqrt[3]{2a^2b^2}.$$

The addition and subtraction of dissimilar radicals can be effected only by means of the signs $+$ and $-$.

MULTIPLICATION AND DIVISION.

158. Let it be required to multiply $\sqrt[7]{a}$ by $\sqrt[7]{b}$, we have $\sqrt[7]{a}\times\sqrt[7]{b}=\sqrt[7]{ab}$; for $\sqrt[7]{a}\times\sqrt[7]{b}$ raised to the seventh power gives ab for the result, and $\sqrt[7]{ab}$ raised to the seventh power gives also ab for the result; whence the seventh powers of these expressions being equal, the expressions themselves must be equal.

The same reasoning may be applied to all similar cases; we have therefore the following rule for the multiplication of radical expressions of the same degree, viz. *Take the product of the quantities under the radical sign, observing to place the result under a sign of the same degree.*

Let it next be required to divide $\sqrt[5]{a}$ by $\sqrt[5]{b}$. In this

case we have $\frac{\sqrt[5]{a}}{\sqrt[5]{b}} = \sqrt[5]{\frac{a}{b}}$; for the expressions $\frac{\sqrt[5]{a}}{\sqrt[5]{b}}$, $\sqrt[5]{\frac{a}{b}}$ being raised to the fifth power give each $\frac{a}{b}$; these expressions are therefore equal.

We have then the following rule for the division of one radical quantity by another of the same degree, viz. *Take the quotient arising from the division of the quantities under the radical sign, recollecting to place it under a sign of the same degree.*

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

159. Let it be required to raise the radical $\sqrt[5]{a^2 b}$ to the third power; we have

$$(\sqrt[5]{a^2 b})^3 = \sqrt[5]{a^2 b} \times \sqrt[5]{a^2 b} \times \sqrt[5]{a^2 b} = \sqrt[5]{a^6 b^3},$$

according to the rule established for multiplication.

Whence to raise a radical quantity to any power; *we raise the quantity placed under the radical sign to the power required, observing to place the result under the same radical sign.*

When the index of the radical is a multiple of the exponent of the power, to which the radical is to be raised, it may be raised to the power required in a more simple manner than by the preceding rule.

Thus let it be required to raise $\sqrt[4]{2a}$ to the second power. The proposed from what has been said art. 155 may be put under the form $\sqrt{\sqrt{2a}}$; but to raise this expression to the second power, it is sufficient to suppress the first radical sign; whence $(\sqrt[4]{2a})^2 = \sqrt{2a}$.

Again, let it be required to raise $\sqrt[9]{5b}$ to the third power. The proposed may be put under the form $\sqrt[3]{\sqrt[3]{5b}}$; whence $(\sqrt[9]{5b})^3 = \sqrt[3]{5b}$.

Whence if the index of the radical is divisible by the exponent of the power, to which the proposed quantity is to be raised, the

operation is performed by dividing the index of the radical by the exponent of the power.

With respect to the extraction of roots, it is evident from the preceding rules, that to extract the root of a radical, *we may extract the root of the quantity placed under the radical sign, the result being left under the same radical sign, or we may multiply the index of the radical by the index of the root to be extracted.*

Thus, $\sqrt[3]{\sqrt[4]{27\,a^3}} = \sqrt[4]{3\,a}$. $\sqrt[3]{\sqrt[4]{3\,c}} = \sqrt[12]{3\,c}$.

160. It follows from the principles established above that, *if we multiply at the same time the index of the radical and the exponents of the quantity placed under the radical sign by the same number, the value of the radical remains the same.*

Thus if we multiply the index of the radical $\sqrt[5]{a^2\,b}$ by 3, we have $\sqrt[15]{a^2 b}$, the third root of the proposed; if then we multiply the exponent of the quantity placed under the radical sign by 3, we have $\sqrt[15]{a^6\,b^3}$ the third power of $\sqrt[15]{a^2\,b}$, the second operation therefore restores the expression to its original value.

161. By means of this last principle, we may reduce two or more radicals of different indices to the same index. Thus let there be the two radicals $\sqrt[3]{2\,a}$, $\sqrt[4]{b^2\,c}$. Multiplying the index and also the exponents of the quantities placed under the radical sign in the first by 4, and in the second by 3, we have for the first $\sqrt[12]{2^4\,a^4}$ or $\sqrt[12]{16\,a^4}$, and for the second $\sqrt[12]{b^6 c^3}$. The proposed are therefore reduced to equivalent expressions having a common index 12.

In like manner the three quantities

$$\sqrt[3]{a\,b^2},\ \sqrt[5]{a^2\,b^3},\ \sqrt[7]{c^4\,d^3}$$

become respectively

$$\sqrt[105]{a^{35}\,b^{70}},\ \sqrt[105]{a^{42}\,c^{63}},\ \sqrt[105]{b^{60}\,c^{45}}$$

having a common index 105.

From what has been done we have the following rule for reducing radical expressions to the same index, viz. ***Multiply at the same time the index belonging to each radical sign, and the exponents of the quantities placed under this sign by the product of the indices belonging to all the other radical signs.***

If the indices of the radicals have common factors, the calculations are rendered more simple by taking for the common index the least number exactly divisible by each of the indices.

162. A quantity, which has no radical sign, may on the same principles be placed under a radical sign; for this purpose, ***we raise the quantity proposed to the power denoted by the index of the radical sign, under which it is to be placed.***

Thus if it be required to put the quantity a^2 under the sign $\sqrt[5]{\ }$, we have for the result $\sqrt[5]{a^{10}}$.

Having reduced radical expressions of whatever degree to the same index, we may then apply to them the rules for multiplication and division laid down above.

EXAMPLES.

1. Multiply $\sqrt[4]{2}$ by $\sqrt[5]{3}$ Ans. $\sqrt[20]{2592}$

2. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$ Ans. $\sqrt[6]{a^3 b^2}$

3. Multiply $\sqrt[3]{a}$ by $\sqrt[4]{b}$ Ans. $\sqrt[12]{a^4 b^3}$

4. Multiply $3a\sqrt[4]{8a^2}$ by $2b\sqrt[4]{4a^2c}$ Ans. $12a^2b\sqrt[4]{2c}$

5. Divide $\sqrt[4]{6}$ by $\sqrt[3]{2}$ Ans. $\frac{1}{2}\sqrt[12]{55296}$

6. Divide $\sqrt[5]{135}$ by $\sqrt{3}$ Ans. $\sqrt[10]{75}$

7. Divide $a\sqrt[3]{b}$ by $\sqrt[4]{b}$ Ans. $a\sqrt[12]{b}$

8. Divide $\sqrt[3]{a^2b^2 + b^4}$ by $\sqrt[3]{\dfrac{a^2 - b^2}{8b}}$

Ans. $2b\sqrt[3]{\dfrac{a^2 + b^2}{a^2 - b^2}}$

THEORY OF EXPONENTS OF ANY NATURE WHATEVER.

163. We have seen, art. 51, that with respect to the same letter, division is performed by subtracting the exponent of the divisor from that of the dividend. The application of this rule to the case, in which the exponent of the divisor is equal to that of the dividend, gives rise to the exponent 0. An expression a^0, in which this exponent is found, is to be regarded, art. 55, as *a symbol equivalent to unity.*

164. The application of the same rule to the case, in which the exponent of the divisor exceeds that of the dividend, gives rise to *negative* exponents. Thus let it be required to divide a^3 by a^5. Subtracting the exponent of the latter from that of the former, we have a^{-2} for the result. But a^3 divided by a^5 is expressed by the fraction $\frac{a^3}{a^5}$; reducing this fraction to its lowest terms, we have $\frac{1}{a^2}$. The expression a^{-2} must therefore be regarded as equivalent to $\frac{1}{a^2}$.

In like manner $\frac{a^m}{a^{m+n}}$ gives by subtracting the exponent of the divisor from that of the dividend a^{-n}; but the fraction $\frac{a^m}{a^{m+n}}$ gives when reduced to its lowest terms $\frac{1}{a^n}$; whence a^{-n} is equivalent to $\frac{1}{a^n}$.

The expression a^{-n} is therefore the symbol of a division, which cannot be performed. *Its true value is the quotient of unity divided by* a *raised to a power denoted by the negative exponent* n.

165. To find the roots of simple quantities, we divide, art. 138, the exponents of the proposed by the index of the root required. The application of this rule to the case, in which the exponents of the proposed are not divisible by

the index of the root, gives rise to *fractional* exponents. Thus let the third root of a be required. Indicating upon the exponent of a the operation required in order to obtain the third root, we have for the result $a^{\frac{1}{3}}$. But we have agreed to indicate the third root by $\sqrt[3]{}$; the expressions $\sqrt[3]{a}$, $a^{\frac{1}{3}}$ are therefore to be regarded as equivalent. In like manner, we have $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

The expression $a^{\frac{m}{n}}$ *is therefore to be regarded as a symbol equivalent to the* n*th root of the* m*th power of* a.

166. The two preceding cases sometimes meet in the same expression. This gives rise to *negative* fractional exponents. Thus let it be required to extract the seventh root of a^3 divided by a^5; we have $\frac{a^3}{a^5} = a^{-2}$, the seventh root of which is $a^{-\frac{2}{7}}$. In like manner the nth root of

$$\frac{a^n}{a^{m+n}} = a^{-\frac{m}{n}}$$

The expression $a^{-\frac{m}{n}}$ is therefore the symbol of a division which cannot be performed, combined with the extraction of a root. *Its true value is the* n*th root of the quotient of unity divided by* a *raised to the* m*th power.*

The expression a^0, a^{-m}, $a^{\frac{m}{n}}$, $a^{-\frac{m}{n}}$, derived in the manner above explained from rules previously established, have become by agreement notations equivalent respectively to 1, $\frac{1}{a^m}$, $\sqrt[n]{a^m}$, $\sqrt[n]{\frac{1}{a^m}}$; we may therefore at pleasure substitute the former of these expressions for the latter and the converse.

167. We proceed to show, that the rules already established for performing the operations of arithmetic upon quantities affected with entire and positive exponents are sufficient for these operations, whatever the exponents may be, with which the quantities are affected.

MULTIPLICATION.

Let it be required to multiply $a^{\frac{3}{5}}$ by $a^{\frac{2}{3}}$. To perform the operation required, it is sufficient *to add the exponents.*

Indeed $a^{\frac{3}{5}} = \sqrt[5]{a^3}$, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$, whence

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = \sqrt[5]{a^3} \times \sqrt[3]{a^2} = \sqrt[15]{a^{19}} = a^{\frac{19}{15}}.$$

But adding the exponents, we have

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = a^{\frac{3}{5}+\frac{2}{3}}\ a^{\frac{19}{15}}$$

the same result as before.

Again, let it be required to multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$; we have $a^{\frac{5}{6}} = \sqrt[6]{a^5}$, and $a^{-\frac{3}{4}} = \sqrt[4]{\frac{1}{a^3}}$

whence $a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = \sqrt[4]{\frac{1}{a^3}} \times \sqrt[6]{a^5} = \sqrt[12]{\frac{1}{a^9}} \times \sqrt[12]{a^{10}} \ldots$

$$= \sqrt[12]{a} = a^{\frac{1}{12}}$$

But adding the exponents of the proposed, we have

$$a^{\frac{5}{6}} \times a^{-\frac{3}{4}} = a^{\frac{5}{6}-\frac{3}{4}} = a^{\frac{1}{12}}$$

the same result as by the former operation.

Let it be required next to multiply $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$; we have $a^{-\frac{m}{n}} = \sqrt[n]{\frac{1}{a^m}}$, $a^{\frac{p}{q}} = \sqrt[q]{a^p}$;

whence $a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = \sqrt[n]{\frac{1}{a^m}} \times \sqrt[q]{a^p}$

$$= \sqrt[nq]{\frac{a^{np}}{a^{mq}}} = \sqrt[nq]{a^{np-mq}} = a^{\frac{np-mq}{nq}}$$

We arrive at the same result by adding the exponents of the proposed.

Indeed $a^{-\frac{m}{n}+\frac{p}{q}} = a^{\frac{np-mq}{nq}}$

To multiply two simple quantities therefore, it is sufficient, whatever the exponents, *to add the exponents of the letters, which are the same in each.*

DIVISION.

168. Whatever the exponents may be, in order to divide one simple quantity by another, we subtract for each letter *the exponent of the divisor from that of the dividend.*

Indeed, since the exponent of each letter in the quotient should be such, that when added to the exponent of the same letter in the divisor, the sum will be equal to the exponent of the dividend, it follows, that the exponent of the quotient should be equal to the difference between that of the divisor and the dividend.

By this rule, we have

$$a^{\frac{2}{3}} \div a^{-\frac{3}{4}} = a^{\frac{2}{3} - (-\frac{3}{4})} = a^{\frac{17}{12}}$$

$$a^{\frac{3}{4}} \div a^{\frac{4}{5}} = a^{\frac{3}{4} - \frac{4}{5}} = a^{-\frac{1}{20}}$$

$$a^{\frac{2}{5}} b^{\frac{3}{4}} \div a^{-\frac{1}{2}} b^{\frac{7}{8}} = a^{\frac{9}{10}} b^{-\frac{1}{8}}$$

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

169. From the rule for multiplication, it follows, that to raise a simple quantity to any power, it is necessary whatever the exponents of the letters may be, to multiply the exponent of each letter by the exponent of the power required.

Thus $a^{-\frac{2}{3}}$ raised to the third power

$$= a^{-\frac{2}{3} + (-\frac{2}{3}) + (-\frac{2}{3})} = a^{-\frac{6}{3}} = a^{-2}$$

Conversely to extract the root of a simple quantity, *we divide the exponent of each letter by the index of the root.*

Thus $\sqrt[3]{a^{-2}} = a^{-\frac{2}{3}}$.

The utility of exponents of the kind, which we are here considering, consists principally in this, that the calculation of quantities affected with these exponents is performed by

the rules already established for quantities affected with entire and positive exponents. The calculation is moreover reduced to operations upon fractions, with which we are already familiar.

170. By means of negative exponents we may give an entire form to fractional expressions. Thus let there be the fraction $\frac{x}{y^2}$, this is the same as $x \times \frac{1}{y^2}$; but $\frac{1}{y^2} = y^{-2}$; whence $\frac{x}{y^2} = x\,y^{-2}$.

171. Fractional and negative exponents enable us to arrange polynomials, which contain radical terms. Thus let it be required to arrange the polynomial

$$2\sqrt{a} + \frac{1}{a} + \sqrt[3]{a^7} + \frac{4}{a\sqrt[3]{a}} + \frac{1}{\sqrt[3]{a^2}} + 2\sqrt[6]{a^6}$$

according to the descending powers of the letter a.

To perform the operation required 1°. we give to the radical quantities fractional exponents. 2°. we reduce to an entire form terms, which have denominators. 3°. we reduce all the exponents of the letter, according to which the arrangement is to be made, to their least common denominator. The proposed may then be arranged according to the powers of the letter required.

In the preceding example we have for the result

$$a^{\frac{14}{6}} + 2\,a^{\frac{5}{6}} + 2\,a^{\frac{3}{6}} + a^{-\frac{4}{6}} + a^{-\frac{6}{6}} + 4\,a^{-\frac{8}{6}}$$

SECTION VI. PROPORTION AND PROGRESSION

172. When two quantities are compared with respect to their magnitude, the result of the comparison is called their *ratio*. In general, there are two different ways, in which the magnitude of two quantities may be compared; 1°. we may wish to determine how much the greater exceeds the less; the result is then obtained by subtraction, and is called the ratio of the quantities *by difference*; 2°. we may wish to determine how often one of the quantities is contained in

the other; the result is then found by division and is called the ratio of the quantities *by quotient.*

Thus the ratio by difference of the quantities a and b is $a-b$, and the ratio by quotient is $\frac{a}{b}$; a and b are the *terms* of the ratio.

The same quantity may be added to, or subtracted from, both terms of a ratio by difference without changing the ratio, for

$$a-b=(a+c)-(b+c)=(a-c)-(b-c)$$

The two terms of a ratio by quotient may be multiplied or divided by the same quantity without changing the ratio, for

$$\frac{a}{b}=\frac{am}{bm}$$

Ratios by difference are sometimes called *arithmetical* ratios, and those by quotient *geometrical* ratios.

173. An expression for two equal ratios is called a *proportion.* If the ratios are by difference, the proportion is called a *proportion by difference.* Thus the equality

$$b-a=d-c$$

is a proportion by difference, and is usually written thus

$$a\,.\,b:c\,.\,d$$

If the ratios are by quotient, the proportion is called a *proportion by quotient.* Thus the equality $\frac{a}{b}=\frac{c}{d}$ is a proportion by quotient, and is usually written thus

$$a:b::c:d$$

The proportions above are read thus, a is to b as c to d. The first and last terms are called the *extremes* of the proportion; the second and third are called the *means*; a is called the *antecedent*, b the *consequent* of the first ratio; c the *antecedent*, d the *consequent* of the second ratio.

Proportion by difference is sometimes called *arithmetical* proportion, that by quotient *geometrical proportion.* Proportion by difference is now, however, more commonly called equidifference, while the term proportion is limited to proportions by quotient.

EQUIDIFFERENCES.

174. Let there be the equidifference $a \,.\, b : c \,.\, d$; this is the same with the equation $b-a=d-c$, from which we deduce $$a+d=b+c$$

Thus in an equidifference *the sum of the extremes is equal to the sum of the means.* This is the leading property of equidifferences.

Reciprocally, let there be four quantities a, b, c, d, such that $a+d=b+c$. From this equation we obtain
$$b-a=d-c \text{ or } a \,.\, b : c \,.\, d$$

Thus *if there be four quantities such, that any two of them give the same sum with the other two, the first are the extremes, the second the means,* or the converse, *of an equidifference.*

Any three terms of an equidifference are sufficient to determine the fourth; thus, from the equidifference $a \,.\, b : c \,.\, d$, we deduce $a=b-d+c$, $b=a+d-c$.

In the equidifference $a \,.\, b : c \,.\, d$, let $c=b$; we have
$$a : b : b \,.\, d.$$

This is called a *continued* equidifference, and b is called a *mean differential or arithmetical mean* between a and d.

From the equidifference $a \,.\, b : b \,.\, d$, we deduce
$$b=\tfrac{1}{2}(a+d);$$
thus *the arithmetical mean between two quantities is equal to half their sum.*

175. In order that an equidifference may exist, it is sufficient, that the sum of the extremes should be equal to the sum of the means; we may therefore make any transposition of the terms of an equidifference, which will not alter the equality between the sum of the extremes and that of the means. The equation $a-b=c-d$ furnishes the eight following equidifferences

$$a \,.\, b : c \,.\, d, \quad a \,.\, c : b \,.\, d, \quad d \,.\, b : c \,.\, a, \quad d \,.\, c : b \,.\, a$$
$$b \,.\, a : d \,.\, c, \quad b \,.\, d : a \,.\, c, \quad c \,.\, d : a \,.\, b, \quad c \,.\, a : d \,.\, b$$

PROPORTION BY QUOTIENT.

176. Let us take the proportion $a : b :: c : d$; this returns to $\frac{b}{a} = \frac{d}{c}$, an equation, which gives

$$a d = b c$$

Thus in a proportion by quotient *the product of the extremes is equal to the product of the means.* This is the fundamental property of proportions.

Reciprocally, let there be four quantities a, b, c, d, such that $a d = b c$; this leads to the equation $\frac{b}{a} = \frac{d}{c}$ or

$$a : b :: c : d.$$

Whence *if four quantities be such, that any two of them give the same product as the remaining two, the first will form the extremes and the second the means*, or the converse, *of a proportion.*

Three terms of a proportion are sufficient to determine the fourth; thus from the proportion $a : b :: c : d$, we deduce

$$a = \frac{b c}{d},\quad b = \frac{a d}{c}, \text{ \&c.}$$

The proportion $a : b :: b : d$, in which the two mean terms are the same, is called a *continued* proportion, and b is called a *mean proportional* between a and d.

From the continued proportion $a : b :: b : d$, we deduce $b^2 = a d$, whence $b = \sqrt{a d}$. Thus *to find a mean proportional between two quantities, we take the square root of their product.*

177. In order that a proportion may exist, it is sufficient, that the product of the extremes should be equal to that of the means. We may therefore make any transposition in the terms of a proportion, which will leave the product of the extremes equal to that of the means. Thus the equation $\frac{b}{a} = \frac{d}{c}$ gives the eight following proportions

$$a : b :: c : d,\quad a : c :: b : d,\quad b : d :: a : c,\quad d : c :: b : a$$
$$b : a :: d : c,\quad c : a :: d : b,\quad d : b :: c : a,\quad c : d :: a : b$$

178. The same quantity m, it is evident, may be added to or subtracted from the equation $\frac{b}{a}=\frac{d}{c}$, so that we have

$$\frac{b}{a}\pm m=\frac{d}{c}\pm m$$

whence $$\frac{b\pm m a}{a}=\frac{d\pm m c}{c}$$

but this last may assume the form

$$\frac{c}{a}=\frac{d\pm m c}{b\pm m a}$$

from which we have the proportion

$$b\pm m a : d\pm m c :: a : c$$

but since $\frac{c}{a}=\frac{d}{b}$, we have also

$$\frac{d}{b}=\frac{d\pm m c}{b\pm m a}$$

from which we have the proportion

$$b\pm m a : d\pm m c :: b : d$$

These two proportions may be enunciated thus; *The first consequent plus or minus its antecedent taken a given number of times, is to the second consequent plus or minus its antecedent taken the same number of times, as the first term is to the third, or as the second is to the fourth.*

179. The expression $\frac{d\pm m c}{b\pm m a}=\frac{c}{a}$ returns to

$$\frac{d+m c}{b+m a}=\frac{c}{a},\ \frac{d-m c}{b-m a}=\frac{c}{a}$$

whence $$\frac{d+m c}{b+m a}=\frac{d-m c}{b-m a}$$

or $$b+m a : d+m c :: b-m a : d-m c$$

or changing the relative places of the means

$$b+m a : b-m a :: d+m c : d-m c$$

whence making $m=1$, we have

$$b+a : b-a :: d+c : d-c$$

a proportion which may be enunciated thus

The sum of the first two terms is to their difference, as the sum of the last two is to their difference.

180. The proportion $a : b :: c : d$ may be written thus,

$$a : c :: b : d,$$

we have then

$$\frac{c}{a} \pm m = \frac{d}{b} \pm m$$

from which we obtain

$$c \pm m a : d \pm m b :: a : b \text{ or } :: c : d$$

whence, *the second antecedent plus or minus the first taken a given number of times is to the second consequent plus or minus the first taken the same number of times, as any one of the antecedents whatever is to its consequent.*

If in the above proportion we make $m = 1$, we have

$$c \pm a : d \pm b :: a : b \text{ or } :: c : d$$

whence $$c + a : c - a :: d + b : d - b$$

Therefore *the sum or difference of the antecedents is to the sum or difference of the consequents, as one antecedent is to its consequent, and the sum of the antecedents is to their difference, as the sum of the consequents is to their difference.*

181. Let there be the series of equal ratios

$$a : b :: c : d :: e : f :: g : h \;\ldots\;.$$

or $$\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \frac{h}{g}$$

Making $\frac{b}{a} = q$, we have

$$\frac{b}{a} = q, \quad \frac{d}{c} = q, \quad \frac{f}{e} = q, \quad \frac{h}{g} = q$$

whence $$b = a q, \; d = c q, \; f = e q, \; h = g q$$

adding these equations member to member, we have

$$b + d + f + h = (a + c + e + g) q$$

whence $$\frac{b + d + f + h}{a + c + e + g} = q = \frac{b}{a}$$

or $$a + c + e + g : b + d + f + h :: a : b$$

whence *in a series of equal ratios, the sum of any number whatever of antecedents, is to the sum of the like number of consequents, as one antecedent is to its consequent.*

182. Let there be the two equations $\frac{b}{a}=\frac{d}{c}$, $\frac{f}{e}=\frac{h}{g}$ multiplying these equations member by member, we have

$$\frac{bf}{ae}=\frac{dh}{cg}$$

or

$$ae : bf :: cg : dh$$

We obtain the same result by multiplying term by term the proportions $a : b :: c : d$, $e : f :: g : h$, this is called multiplying the proportions *in order*; it follows then, *that if two proportions be multiplied in order, the results will be proportional.*

It will be seen also, that if two proportions be divided term by term or in order, the quotients will be proportional.

If in the equation $\frac{b}{a}=\frac{d}{c}$ we raise both members to the mth power, we have

$$\frac{b^m}{a^m}=\frac{d^m}{c^m}$$

which gives $a^m : b^m :: c^m : d^m$.

It follows therefore *that the squares, cubes, and in general the similar powers of four proportional quantities are also proportional.*

In like manner it may be shown, *that the roots of the same degree of four proportional quantities are also proportional.*

PROGRESSIONS.

183. A series of quantities increasing or decreasing by a constant difference is called an *arithmetical progression* or *progression by difference.* The constant difference is called the *ratio* of the progression.

Thus let there be the two following series

1, 4, 7, 10, 13, 16
60, 56, 52, 48, 44, 40

the first is called an *increasing* progression, the ratio of which

is 3 ; the second is called a *decreasing* progression, the ratio of which is 4.

To indicate that the quantities a, b, c, d . . . form a progression by difference, we write them thus

$$\div a \,.\, b \,.\, c \,.\, d \,.\,.\,.$$

A progression by difference, it will readily be perceived, is simply a series of continued equidifferences. Each term therefore is at once antecedent and consequent with the exception of the first term, which is only an antecedent and of the last, which is only a consequent. The progression $\div a \,.\, b \,.\, c \,.\, d$ is ennunciated thus, a is to b as b to c, as c to d, &c.

184. Let us take the increasing progression

$$\div a \,.\, b \,.\, c \,.\, d \,.\, h \,.\,.\,.$$

and let d represent the ratio.

From the nature of the progression, we have, it is evident

$$\begin{aligned} b &= a + d \\ c &= a + 2d \\ d &= a + 3d \\ &\ldots \end{aligned}$$

from which it is readily inferred, *that a term of any rank whatever is equal to the first term plus as many times the ratio, as there are units in the number of the preceding terms.*

Let L represent a term of any rank whatever, and let n denote the number, which marks the place of this term ; we have from what has been said

$$L = a + (n - 1)\,d$$

This expression for L is called the *general term* of the series. If the series were decreasing, we should have, as it is easy to see, for the general term

$$L = a - (n - 1)\,d$$

By means of the above formulas we may find any term of a progression by difference, when the first term, the number of the term required, and the ratio are given.

Thus, let it be required to find the 50th term of the progression $\div 1 . 4 . 7 . . . .$, we have by the first formula

$$L = 1 + (50 - 1)3 = 148.$$

Again, let it be required to find the 40th term of the progression $\div 5 . 3 . 1 . . .$, we have by the second formula

$$L = 5 - (40 - 1)2 = -73$$

185. The first and last terms of a progression are called the *extremes*; if the number of terms be odd, the middle term is called the *mean*; if the number of terms be even, the two terms having the same number of terms on each opposite side are called the *means*.

Let us take the general progression $\div a . b . c . . . h . k . l$; from the nature of the progression, we have

$$b - a = l - k$$

whence $\quad b + k = a + l$

so also $\quad c - b = k - h$

whence $\quad c + h = b + k = a + l$

. . . .

from which we infer that in a progression by difference, *the sum of any two terms taken at equal distances from the extremes is equal to the sum of the extremes.*

Let S represent the sum of all the terms in the progression $\div a . b . c . . . . h . k . l$. Writing this progression in an inverse order below itself, we have

$$S = a + b + c \, ... \, h . k . l$$
$$S = l + h + k \, ... \, c . b . a$$

adding these equations member to member and uniting the corresponding terms, we have

$$2S = (a + l) + (b + k) \, ... + (h + c) + (k + b) + (l + a)$$

but the parts $b + h$, $h + c$ .... are equal each to $a + l$; the number of these parts moreover is the same, it is evident, with the number of the terms in the progression; designating then this number by n, we have

$$2S = n(a + l)$$

whence $$S=\frac{n(a+l)}{2}$$

By means of this formula we find the sum of all the terms, when the first term, the last term and the number of the terms are given.

186. The equations $L=a+(n-1)d$, $S=\frac{n(a+l)}{2}$ furnish us with the means of resolving the following general problem, viz. *Any three of the five quantities* a, d, n, l *and* s, *which enter into a progression by difference, being given, to determine the remaining two.*

This general problem resolves itself into as many particular problems as there are combinations of 5 letters taken 2 and 2 or 3 and 3 at a time. The number will therefore be 10. See the enunciations below.

Let there be given	1°. a, d, n	to find	l and s
	2°. a, d, l	. . .	n and s
	3°. a, d, s	. . .	n and l
	4°. a, n, l	. . .	d and s
	5°. a, n, s	. . .	d and l
	6°. a, l, s	. . .	d and n
	7°. d, n, l	. . .	a and s
	8°. d, n, s	. . .	a and l
	9°. d, l, s	. . .	a and n
	10°. n, l, s	. . .	a and d

Of these problems we shall resolve the fourth only, leaving the rest as an exercise for the learner. Let there be given then a, n and l to find s and d. The expression for s we have already in the formula $S=\frac{n(a+l)}{2}$; with respect to d the formula $L=a+(n-1)d$ gives $d=\frac{l-a}{n-1}$.

187. This expression for d enables us to resolve the following problem, viz. *to insert between two quantities* b *and* c m *mean differentials*, that is to say, quantities, which comprised between b and c, will form with them a progression by difference.

To resolve this problem, it will be sufficient to determine the ratio of the progression required. For this we have given the first term b, the last term c, and the number of terms $m+2$. Substituting therefore c, b and $m+2$ for l, a, and n in the above expression for d, viz. $d=\frac{l-a}{n-1}$, the ratio required will be $\frac{c-b}{m+2-1}=\frac{c-b}{m+1}$, that is, to find the ratio sought, *we divide the difference of the two numbers* b *and* c *by the number of terms to be inserted plus* 1.

Let it be required, for example, to insert 11 mean differentials between 17 and 77.

Here $$d=\frac{77-17}{12}=5$$

The progression required will therefore be

$$\div\ 17.\ 22.\ 27.\ 32\ \ldots\ldots\ 72.\ 77$$

It will readily be inferred from what has been done, that, *if between the terms of a progression by difference, taken two and two, we insert the same number of mean differentials, the terms of this progression together with the mean differentials inserted will form a progression by difference.*

EXAMPLES IN PROGRESSION BY DIFFERENCE.

1. A number consisting of 3 digits, which are in arithmetical progression, being divided by the sum of its digits gives a quotient 48; and if 198 be subtracted from it the digits will be inverted. Required the number.

Let $x=$ the second digit and y the common difference, the three digits will then be expressed by $x+y$, x, $x-y$.

Resolving the question we obtain $x=3$, and the number required is 432.

2. Four numbers are in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is equal to 32. What are the numbers?

Let $2y=$ the common difference, and $x+3y$, $x+y$, $x-y$, $x-3y$ be the numbers.

Resolving the question, we obtain for the numbers sought, 11, 9, 7 and 5.

3. A traveller sets out for a certain place, and travels 1 mile the first day, 2 the second and so on. In 5 days afterwards another sets out and travels 12 miles a day. How long and how far must he travel to overtake the first?

Let $x =$ the number of days; then $x + 5 =$ the number of days the first travels, and $(x + 6)\frac{x+5}{2} =$ the distance he travels.

Resolving the question, we obtain $x = 3$ or 10.

4. There are three numbers in arithmetical progression, whose sum is 21; and the sum of the first and second is to the sum of the second and third as 3 to 4. Required the numbers. Ans. 5, 7, 9.

5. From two towns, which were 168 miles distant, two persons A and B set out to meet each other; A went 3 miles the first day, 5 the next, and so on; B went 4 miles the first day, 6 the next, and so on. In how many days did they meet? Ans. 8.

6. There are four numbers in arithmetical progression, whose sum is 28, and their continued product is 585. Required the numbers. Ans. 1, 5, 9, 13.

7. A and B, 165 miles distant from each other, set out with a design to meet; A travels 1 mile the first day, 2 the second, and so on; B travels 20 miles the first day, 18 the second, and so on. How soon will they meet?

Ans. They will meet in 10, and also in 33 days.

8. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Ans. 14, 10, 6, 2 or -14, -10, -6, -2.

9. The product of five numbers in arithmetical progression is 945, and their sum is 25. Required the numbers.

Ans. 9, 7, 5, 3, 1.

10. A regiment of men was just sufficient to form an equilateral wedge. It was afterwards doubled, but was still

found to want 385 men to complete a square containing 5 more men in a side, than in a side of the wedge. How many did the regiment at first contain? Ans. 850. 820

11. After A, who travelled at the rate of 4 miles an hour, had been set out two hours and three quarters, B set out to overtake him, and in order thereto went four miles and a half the first hour, four and three quarters the second, five the third, and so on, gaining a quarter of a mile every hour. In how many hours would he overtake A? Ans. 8.

PROGRESSION BY QUOTIENT.

188. A series of quantities such, that if any term be divided by the one which precedes it, the quotient is the same in whatever part of the series the two terms be taken, is called *a geometrical progression or progression by quotient.*

The constant quotient is called the *ratio* of the progression. If the series is increasing, the ratio will be greater than unity, if decreasing, the ratio will be less than unity.

The following series are examples of this kind of progression,

$$3 \;.\; 6 \;.\; 12 \;.\; 24 \;.\; 48 \;.\; 96$$
$$64 \;.\; 16 \;.\; 4 \;.\; 1 \;.\; \tfrac{1}{4} \;.\; \tfrac{1}{16}$$

In the first the ratio is 2, in the second $\frac{1}{4}$. A progession by quotient, it will readily be perceived, is simply a series of equal ratios by quotient, in which *each term is at once antecedent and consequent, with the exception of the first, which is only an antecedent, and the last, which is only a consequent.*

To indicate that the quantities a, b, c, d . . . form a progression by quotient, we write them thus

$$\div\!\div a : b : c : d : \ldots$$

The progression is enunciated thus, a to b as b to c as c, &c.

189. Let us take the general progression

$$\div\!\div a : b : c : d \ldots$$

and let the ratio be represented by q; from the nature of the progression, we have, it is evident,

$$b = a\,q,\; c = b\,q = a\,q^2,\; d = c\,q = a\,q^3$$

from which it will be readily inferred, that *a term of any rank whatever is equal to the first term multiplied by the ratio raised to a power, the exponent of which is one less than the number, which marks the place of this term.*

Let L designate any term whatever of the progression, and let n represent the number of this term; from what has been said, we have

$$L = a q^{n-1}.$$

This is called the *general term* of the progression. By means of it we may find any term required, when the first term and the ratio are given.

Thus let it be required to find the 8th term of the progression $\div\!\div 2 : 6 : 18 \ldots$ in this case, we have

$$L = 2 \times 3^7 = 4374.$$

In like manner if it be required to find the 12th term of the progression $\div\!\div 64 : 16 : 4 : 1 : \frac{1}{4}, \ldots$, we have

$$L = 64 \left(\frac{1}{4}\right)^{11} = \frac{1}{65536}$$

190. Resuming the general progression

$$\div\!\div a : b : c : d \ldots k : l,$$

we have from the nature of the progression the series of equations,

$$b = a q, \ c = b q, \ d = c q \ \ldots \ l = k q$$

adding these equations member to member, we have

$$b + c + d + \ldots l = (a + b + c + \ldots k)\, q \quad (1)$$

Let S represent the sum of all the terms, we have

$$b + c + d + \ldots \ldots l = S - a$$
$$a + b + c + \ldots \ldots k = S - l$$

whence by substitution in equation (1), we have

$$S - a = q\,(S - l)$$

and by consequence $\quad S = \dfrac{q\,l - a}{q - 1}$

By means of this formula we may obtain the sum of any number of terms of a progression by quotient; for this pur-

pose, *we multiply the last term by the ratio, subtract the first term from this product, and divide the remainder by the ratio diminished by unity.*

Let it be required to find the sum of the first 4 terms of the progression $\div\!\!\div\, 2 : 6 : 18 : 54 : 162$, we have

$$S = \frac{54 \times 3 - 2}{3 - 1} = 80$$

191. When the progression is decreasing, that is, when q is less than 1, it will be more convenient to put the above expression for S under the form $S = \frac{a - lq}{1 - q}$; since in this case the two terms of the fraction will be positive.

Let it be required to find the sum of the 12 first terms of the progression $\div\!\!\div\, 64 : 16 : 4 : 1 : \frac{1}{4} \ldots \frac{1}{65536}$.

We have $S = \frac{a - lq}{1 - q} = \frac{64 - \frac{1}{65536} \cdot \frac{1}{4}}{1 - \frac{1}{4}} = 85 + \frac{65535}{196608}$.

192. If in the formulas for S we substitute for l its value, viz. $l = a q^{n-1}$, we have

$$S = \frac{aq^n - a}{q - 1}, \quad S = \frac{a - aq^n}{1 - q}$$

formulas, by means of which, we obtain the sum of any number of terms of a progression, when the number of terms, the first term, and ratio are given.

Thus to find the sum of the first 8 terms of the progression $\div\!\!\div\, 2 : 6 : 18 : 54 \ldots$ we have

$$S = \frac{aq^n - a}{q - 1} = \frac{2 \times 3^8 - 2}{3 - 1} = 6560$$

In the same manner, we have for the sum of the 12 first terms of the progression $\div\!\!\div\, 64 : 16 : 4 \ldots$

$$S = \frac{a - aq^n}{1 - q} = \frac{64 - 64.\left(\frac{1}{4}\right)^{12}}{1 - \frac{1}{4}} = 85 + \frac{65535}{196608}$$

INFINITE PROGRESSIONS BY QUOTIENT.

193. Let there be the decreasing progression

$$\div\!\!\div\, a : b : c : d \ldots$$

consisting of an infinite number of terms. The formula for the sum of any number of terms, viz. $S=\frac{a-aq^n}{1-q}$ may be put under the form

$$S=\frac{a}{1-q}-\frac{a}{1-q}\cdot q^n$$

But since the progression is decreasing q is a fraction; q^n is also a fraction; hence as the number n becomes greater or as we take more terms, the expression $\frac{a}{1-q}\cdot q^n$ becomes smaller, and the value of S approaches nearer to $\frac{a}{1-q}$. If then we suppose n greater than any assignable quantity or *infinite*, $\frac{a}{1-q}\cdot q^n$ will be less than any assignable quantity or 0, and $\frac{a}{1-q}$ will in this case represent the true value of the series.

We conclude therefore that the sum of the terms of a decreasing progression, in which the number of terms is infinite, has for its expression $S=\frac{a}{1-q}$, q being the ratio of the progression and a the first term.

194. Strictly speaking the quantity $\frac{a}{1-q}$, is the *limit*, which the sum of a decreasing progression can never surpass, but to which it continually approximates as we take more terms.

Let there be, for example, the progression

$$\div\, 1 : \tfrac{1}{2} : \tfrac{1}{4} : \tfrac{1}{8} \;.\;.\;.\;.,$$

we have $\qquad a=1,\; q=\frac{1}{2}$, whence

$$S=\frac{a}{1-q}-\frac{a}{1-q}\cdot q^n=\frac{2}{2-1}-\frac{2}{2-1}\times\left(\frac{1}{2}\right)^n=\frac{2}{2-1}-\frac{1}{2^{n-1}}$$

Here the greater the value of n or the more terms we take, the less is the fraction $\frac{1}{2^{n-1}}$, and the nearer the sum of the series approaches to 2. If the number of terms be

considered infinite, the fraction $\frac{1}{2^{n-1}}$ will be less than any assignable quantity or zero, and the sum of the series will be equal to 2.

Strictly speaking, however, 2 is the limit, which the sum of the proposed series can never surpass, but to which it constantly approximates as we take more terms.

Thus let the number of terms be 1, 2, 3, 4 . . . . successively, we have

$$\begin{aligned}
&1 &&= 2 - 1\\
&1 + \tfrac{1}{2} &&= 2 - \tfrac{1}{2}\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} &&= 2 - \tfrac{1}{4}\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= 2 - \tfrac{1}{8}\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} &&= 2 - \tfrac{1}{16}
\end{aligned}$$

Here the more terms we take, the nearer the sum of the progression will approach to 2, from which it may be made to differ by a quantity as small as we please, though strictly speaking, it can never become equal to 2.

195. When the series is increasing, that is, when q is greater than unity, the expression $S = \frac{a}{1-q}$ cannot be considered as the limit, which the sum of the series can never surpass. For the sum of a determinate number n of terms being $S = \frac{a}{1-q} - \frac{a q^n}{1-q}$, it is evident, that $\frac{a q^n}{1-q}$ will increase more and more numerically in proportion as n increases; by consequence the more terms we take, the more will the sum of the terms differ numerically from $\frac{a}{1-q}$. In this case $\frac{a}{1-q}$ is merely the algebraic expression, which by its development gives rise to the series

$$a + a q + a q^2 + a q^3 + \ldots .$$

Indeed if we perform upon a the division indicated, we have

$$\frac{a}{1-q} = a + a q + a q^2 + a q^3 + \ldots .$$

196. In the above expression let $a=1$, $q=2$, we have

$$\frac{1}{1-2} \text{ or } -1 = 1+2+4+8+16+\ .\ .\ .$$

an equation, in which the first member is negative, while the second is positive, and greater in proportion as q is greater.

In order to interpret this result we observe, that if in the equation $\frac{a}{1-q} = a + a\,q + a\,q^2\ .\ .\ .\ .$ we stop the series at any particular term, it is necessary, in order to preserve the equality of the two members, to complete the quotient by annexing to it the fraction, which remains. If, for example, we stop the series at the fourth term $a\,q^3$, we shall have by completing the quotient

$$\frac{a}{1-q} = a + a\,q + a\,q^2 + a\,q^3 + \frac{a\,q^4}{1-q}$$

an equation which is exact. Indeed if in this equation, we make $a=1$, $q=2$, we have

$$-1 = 1+2+4+8+\frac{16}{-1}$$

from which we obtain $-1=-1$.

197. The equations $l = a\,q^{n-1}$, $S = \frac{l\,q-a}{q-1}$ contain all the relations of the five quantities a,l,q,n and S; we have then the general problem, *any three of the five quantities* a, l, q, n *and S being given, to find the remaining two.* This general problem gives rise to ten particular problems, the enunciations of which will not differ from those relative to progressions by difference art. 186 with the exception, that the ratio is here expressed by the letter q instead of d.

198. From the formula $l = a\,q^{n-1}$, we obtain

$$q = \sqrt[n-1]{\frac{l}{a}}$$

This expression for q enables us to resolve the following problem, viz. *to insert between two given uumbers* b *and* c m *mean proportionals*, that is to say, a number m of *quantities*,

which comprised between b *and* c *will form with them a progression by quotient.*

To resolve this problem it will be sufficient to determine the ratio of the progression required; for this we have given the first term b, the last term c and the number of terms $m+2$.

Substituting therefore b, c and $m+2$ for a, l and n in the above expression for q, we have for the ratio of the required progression

$$q = \sqrt[m+1]{\frac{c}{b}}$$

whence to find the ratio sought, *we divide the given numbers* b *and* c, *one by the other, and extract the root of the quotient to the degree marked by the number of terms to be inserted plus one.*

Let it be required to insert six *mean* proportionals between the numbers 3 and 384. Here $m=6$, we have therefore

$$q = \sqrt[7]{\frac{384}{3}} = \sqrt[7]{128} = 2$$

The progression required is therefore

$$\div 3 : 6 : 12 : 24 : 48 : 96 : 192 : 384$$

From what has been done, it will be easy to see, that *if between the terms of a progression by quotient taken two and two, we insert the same number of mean proportionals, the partial progressions thus formed will together form a progression by quotient.*

199. Of the ten particular problems, which may be proposed upon progressions by quotient, four only can be resolved by principles thus far laid down. Below we have the enunciation of these problems with their answers.

1°. a, q, n being given to find l and S

$$l = a\,q^{n-1}, \quad S = \frac{a\,(q^n - 1)}{q - 1}$$

2°. a, n, l being given to find q and S

$$q = \sqrt[n-1]{\frac{l}{a}}, \quad S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$$

3°. q, n, l being given to find a and S

$$a = \frac{l}{q^{n-1}}, \quad S = \frac{l(q^n - 1)}{q^{n-1}(q-1)}$$

4°. q, n, S being given to find a and l

$$a = \frac{S(q-1)}{q^n - 1}, \quad l = \frac{Sq^{n-1}(q-1)}{q^n - 1}$$

Of the remaining problems, two, viz. those in which a and q, l and q are the unknown quantities, depend upon the resolution of equations of a degree superior to the second. The other four depend upon the resolution of an equation of a nature altogether different from any which we have yet seen, viz. upon an equatiou of the form $a^x = b$, in which the exponent is the unknown quantity.

EXAMPLES IN PROGRESSION BY QUOTIENT.

1. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15. Also the difference of the squares of the greatest and least is to the sum of the squares of all the three numbers as 5 to 7. Required the numbers.

Let x, xy, xy^2 be the numbers; then by the question, we have

$$xy^2 - x = 15$$

and $$7(x^2y^4 - x^2) = 5(x^2y^4 + x^2y^2 + x^2)$$

or by division $$7(y^4 - 1) = 5(y^4 + y^2 + 1)$$

or performing the operations indicated transposing and reducing

$$y^4 - \frac{5}{2}y^2 = 6$$

whence resolving this last, we have

$$y^2 = 4 \text{ and } y = 2$$

Substituting next for y its value in the first equation, we obtain $x = 5$. The numbers required are therefore 5, 10 and 20.

2. The sum of three numbers in geometrical progression is 13, and the product of the mean, and the sum of the extremes is 30. Required the numbers.

Let the numbers be $\frac{x}{y}$, x and $x\,y$; then by the question, we have

$$\frac{x}{y}+x+x\,y=13$$

and
$$\left(\frac{x}{y}+x\,y\right)x=30$$

By transposition in the first equation, we have

$$\frac{x}{y}+x\,y=13-x$$

whence by substitution in the second, we obtain

$$(13\,x-x)\,x=30$$

whence
$$x^2-13\,x=-30$$

from which we deduce

$$x=10,\ x=3$$

Substituting the value $x=3$ in the first equation, we obtain $y=3$ or $\frac{1}{3}$, and the numbers sought are 1, 3, 9.

3. A gentleman divided £210 among three servants in geometrical progression; the first had £90 more than the last. How much had each?

4. There are three numbers in geometrical progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

5. The sum of three numbers in geometrical progression is 35, and the mean term is to the difference of the extremes as 2 to 3. Required the numbers. Ans. 5, 10, 20.

6. The sum of £14 was divided between three persons, whose shares were in geometrical progression; the sum of the shares of the first and second was to the sum of the shares of the second and third as 1 to 2. Required the shares. Ans. 2, 4, 8.

THEORY OF CONTINUED FRACTIONS.

200. In order to form a more exact idea of a fraction, the terms of which are large numbers and prime to each other, we seek approximate values of this fraction, which are expressed in more simple numbers.

Let there be, for example, the fraction $\frac{159}{493}$. Dividing both terms of this fraction by the numerator, an operation, which will not change its value, it becomes $\cfrac{1}{3+\cfrac{16}{159}}$

If then we neglect, for the moment, the fraction $\frac{16}{159}$ in this expression, the result $\frac{1}{3}$ will be greater than the proposed, since the denominator has been diminished.

On the other hand, if instead of neglecting the fraction $\frac{16}{159}$, we substitute 1 for it, the result $\frac{1}{4}$ will be less than the proposed, since the denominator has been increased.

We conclude therefore that the fraction $\frac{159}{493}$ is comprised between $\frac{1}{3}$ and $\frac{1}{4}$, we are thus enabled to form a very exact idea of its value.

If a greater degree of approximation be required, we have only to operate upon $\frac{16}{159}$ in the same manner as we have already done upon $\frac{159}{493}$, we have thus

$$\frac{16}{159}=\cfrac{1}{9+\cfrac{15}{16}}$$

and the proposed fraction becomes

$$\cfrac{1}{3+\cfrac{1}{9+\cfrac{15}{16}}}$$

If we neglect $\frac{15}{16}$, $\frac{1}{9}$ is greater than $\frac{16}{159}$, it follows therefore that $\cfrac{1}{3+\cfrac{1}{9}}$ is less than $\frac{159}{493}$; but $\cfrac{1}{3+\cfrac{1}{9}}$ becomes $\cfrac{1}{\frac{28}{9}}$ or $\frac{9}{28}$; thus the proposed is comprised between $\frac{1}{3}$ and $\frac{9}{28}$. The difference between these two fractions is $\frac{1}{84}$; the error therefore committed in taking $\frac{1}{3}$ or $\frac{9}{28}$ for the value of the proposed fraction is less than $\frac{1}{84}$.

To attain a still greater degree of approximation, we operate in the same manner upon $\frac{15}{16}$; thus we have

$$\frac{15}{16}=\cfrac{1}{1+\cfrac{1}{15}}$$

and the proposed fraction becomes

$$\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{15}}}}$$

Neglecting $\frac{1}{15}$, the fraction $\frac{1}{1}$ or 1 is greater than $\frac{15}{16}$; hence $\cfrac{1}{9+\cfrac{1}{1}}$ or $\frac{1}{10}$ is less than $\frac{16}{159}$; therefore $\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{1}}}$

or $\frac{10}{31}$ is greater than $\frac{159}{493}$; thus the proposed is comprised between $\frac{9}{28}$ and $\frac{10}{31}$. The difference between these two fractions is $\frac{1}{868}$ the error committed therefore in taking either $\frac{9}{28}$ or $\frac{10}{31}$ for the value of the proposed is less than $\frac{1}{868}$.

The expression $\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{15}}}}$ is called a *continued fraction*. We understand therefore by a continued fraction *a fraction, which has unity for its numerator, and for its denominator an entire number plus a fraction, which fraction has also unity for its numerator and for its denominator an entire number plus a fraction, and thus in order*.

It sometimes happens, that the proposed fractional number is greater than unity; to generalize therefore the above definition, we say, that a continued fraction is *an expression composed of an entire number plus a fraction, which has unity for its numerator, and for its denominator an entire number plus a fraction, &c.*

If we examine the above process for converting $\frac{159}{493}$ into a continued fraction, it will be perceived, that we have divided first 493 by 159, which gives 3 for a quotient and a remainder 16; we then divide 159 by 16, which gives 9 for a quotient and a remainder 15; we next divide 16 by 15, which gives 1 for a quotient and a remainder 1; from which we readily infer the following rule for converting a fraction or fractional number into a continued fraction, viz.

Apply to the two terms of the fraction the process for finding their greatest common divisor; pursue the operation until a remainder is obtained equal to 0; *the successive quotients, thus ob-*

tained will be the denominators of the fractions, which constitute the continued fraction.

If the proposed be greater than unity the first quotient will be the entire part, which enters into the expression of the continued fraction.

EXAMPLES. Let the fractions $\frac{73}{137}$, $\frac{829}{347}$ be converted into continued fractions.

201. From what has been said a continued fraction may be represented generally by the expression

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\ldots}}}$$

$a, b, c, d \ldots$ being entire and positive numbers. The fractional number, to which this expression is equivalent, may moreover be represented by x.

The fractions $\frac{1}{b}, \frac{1}{c}, \frac{1}{d} \ldots$, the assemblage of which constitutes the continued fraction, are called *integrant fractions*. The denominators $b, c, d \ldots$ are called *incomplete quotients*, since b, for example, is only the entire part of the number expressed by $b+\cfrac{1}{c+\cfrac{1}{d+\ldots}}$ and c only the entire part of the number expressed by $c+\cfrac{1}{d+\ldots}$ and thus in order. Conversely the expressions $b+\cfrac{1}{c+\cfrac{1}{d\ldots}}$

$c+\cfrac{1}{d+\ldots}$ are called *complete quotients.*

The results obtained by converting successively into a single fractional number each of the expressions

$$a+\frac{1}{b},\quad a+\cfrac{1}{b+\cfrac{1}{c}}\quad \&c.$$

are called *reductions*.

202. The formation of these reductions is subject to a very simple law, which we shall now develop.

The first is a, which may be put under the form $\frac{a}{1}$; the second is $a+\frac{1}{b}$, or reducing the whole expression to a fraction $\frac{ab+1}{b}$. To form the third, represented by

$$a+\cfrac{1}{b+\cfrac{1}{c}},$$

it will be sufficient to substitute $b+\frac{1}{c}$ for b in the second; making this substitution, we have

$$a+\cfrac{1}{b+\cfrac{1}{c}}=\frac{a\left(b+\frac{1}{c}\right)+1}{b+\frac{1}{c}}=\frac{(ab+1)c+a}{bc+1}$$

To form the fourth reduction, it will be sufficient to substitute $c+\frac{1}{d}$ for c in the third; thus we have

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}=\frac{(ab+1)\left(c+\frac{1}{d}\right)+a}{b\left(c+\frac{1}{d}\right)+1}=\ldots$$

$$\frac{[(ab+1)c+a]d+ab+1}{(bc+1)d+b}$$

The first four reductions therefore will be

$$\frac{a}{1},\quad \frac{ab+1}{b},\quad \frac{(ab+1)c+a}{bc+1},\quad \frac{[(ab+1)c+a]d+ab+1}{(bc+1)d+b}$$

Without proceeding further, it will be perceived, that the numerator of the third reduction is formed by multiplying the numerator of the second by the third incomplete quotient c, and adding to this product the numerator of the first reduction. With respect to the denominator, it is formed in the same manner by means of the denominators of the second and first reductions.

In like manner, the numerator and denominator of the fourth reduction is formed, it will be perceived, by multiplying respectively the two terms of the third reduction by the fourth incomplete quotient d and adding to the two products respectively the two terms of the second reduction.

From what has been done it will be readily inferred, that the above law of formation for the third and fourth reductions should be extended to those, which follow. To demonstrate this law, however, in a rigorous manner, we shall show that if it be true in regard to any three successive reductions whatever, it will be true for the reduction, which follows; thus this law being already found true for the first three reductions will be true for the fourth, and being true for the second third and fourth, it will be true for the fifth, and thus in order; it will therefore be general.

Let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$ be any three successive reductions whatever; let r be the incomplete quotient, at which we stop in order to form the reduction $\frac{R}{R'}$, and let it be supposed, that we have

$$\frac{R}{R'}=\frac{Qr+P}{Q'r+P'}$$

Let $\frac{1}{s}$ be the integrant fraction, which follows r, and let $\frac{S}{S'}$ be the corresponding reduction. In order to form this reduction, it is sufficient to substitute in the expression for $\frac{R}{R'}$, $r+\frac{1}{s}$ instead of r; making this substitution, we

have $\frac{S}{S'}=\frac{Q\left(r+\frac{1}{s}\right)+P}{Q'\left(r+\frac{1}{s}\right)+P'}=\frac{(Qr+P)s+Q}{(Q'r+P')s+Q'}=\frac{Rs+Q}{R's+Q'}$

We see therefore that $\frac{S}{S'}$ is formed from the two preceding reductions according to the law enunciated above. This law is therefore general; whence *To form the numerator of any reduction whatever, we multiply the numerator of the preceding reduction by the incomplete quotient, which corresponds to it, and add to the product the numerator of the reduction, which precedes by two ranks the one which we wish to form; the denominator is formed by the same law by means of the two preceding denominators.*

When the number reduced to a continued fraction is less than unity, we substitute $\frac{0}{1}$ instead of a in order to apply the law, which supposes necessarily, that we have already the first two reductions.

Let it be proposed to find the successive reductions of the continued fraction

$$\frac{65}{149}=\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{2}}}}}}$$

The first two reductions being $\frac{0}{1}, \frac{1}{2}$, we have for those which follow

$$\frac{3}{7}, \frac{7}{16}, \frac{17}{39}, \frac{24}{55}, \frac{65}{149}$$

In like manner we have for the several reductions of the continued fraction arising from $\frac{829}{347}$,

$$\frac{2}{1},\ \frac{5}{2},\ \frac{7}{3},\ \frac{12}{5},\ \frac{43}{18},\ \frac{829}{347}$$

So also the fraction $\frac{29}{77}$ being converted into a continued fraction gives the following reductions, viz.

$$\frac{0}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{5},\ \frac{3}{8},\ \frac{29}{77}$$

203. In the preceding examples the successive reductions, it will be perceived, are alternately less and greater than the whole continued fraction, and they approximate this fraction nearer and nearer.

Let x denote the whole continued fraction; let us take the two consecutive reductions $\frac{P}{P'}$, $\frac{Q}{Q'}$ of any rank whatever, and let it be proposed to estimate the difference

$$x - \frac{P}{P'},\quad x - \frac{Q}{Q'}.$$

In the expression for the following reduction, viz. $\frac{R}{R'}$ or $\frac{Qr+P}{Q'r+P'}$ let us substitute for r the complete quotient, of which r is only the entire part, viz. $r + \cfrac{1}{s + \cfrac{1}{t}}$ and let us represent this last by y, we shall thus form the reduction of the whole continued fraction, and by consequence we shall have

$$y > 1 \text{ and } x = \frac{Qy+P}{Q'y+P'}.$$

from this equation, we obtain

$$xQ'y + xP' = Qy + P$$

whence

$$y(xQ' - Q) = P - xP'$$

or

$$Q'y\left(x - \frac{Q}{Q'}\right) = P'\left(\frac{P}{P'} - x\right)$$

but since the quantities $Q'y$ and P' are positive, it follows that $x-\frac{Q}{Q'}$ and $\frac{P}{P'}-x$ must have the same sign; if then $\frac{Q}{Q'}$ be less than x, $\frac{P}{P'}$ must be greater than x and the converse; thus the value of x, or the value of the whole continued fraction is comprised between $\frac{P}{P'}$ and $\frac{Q}{Q'}$ any two consecutive reductions whatever.

Moreover $x-\frac{Q}{Q'}$ is less than $\frac{P}{P'}-x$; for, since we have $Q'>P'$ and $y>1$, we have $Q'y>P'$, and by consequence

$$x-\frac{Q}{Q'}<\frac{P}{P'}-x$$

Thus any reduction whatever $\frac{Q}{Q'}$ approaches more nearly the value of x, than any preceding reduction.

Since the first reduction is always less than the whole continued fraction x, it follows *that the reductions of an even rank are greater than the whole continued fraction, and those of an odd rank are less.* And since these reductions approach nearer and nearer the value of x, the reductions of an odd rank must go on increasing, while those of an even rank decrease. The reductions therefore form two *series*, the terms of which approach nearer and nearer the value of the whole continued fraction.

204. Let us take the three reductions $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$ of any rank whatever, we have

$$\frac{R}{R'}-\frac{Q}{Q'}=\frac{RQ'-QR'}{R'Q'}$$

but since $R=Qr+P$, $R'=Q'r+P'$, we have by substitution

$$\frac{R}{R'}-\frac{Q}{Q'}=\frac{(Qr+P)Q'-Q(Q'r+P')}{R'Q'}$$

or $$\frac{R}{R'}-\frac{Q}{Q'}=\frac{Q'P-QP'}{R'Q'}$$

but $$\frac{Q}{Q'}-\frac{P}{P'}=\frac{QP'-Q'P}{Q'P'}$$

It follows therefore that, *the numerators of two consecutive differences are equal and of contrary signs.*

If in the general continued fraction we take the difference of the first and second reductions, we have

$$\frac{ab+1}{b}-\frac{a}{1}=\frac{1}{b}$$

The difference between any two consecutive reductions whatever will have unity therefore for its numerator. Since moreover the numerator of the first difference is $+1$ that of the second will be -1, that of the third $+1$, and thus in order. In general, *the numerator of any difference whatever will be $+1$, if the second of the reductions under consideration is of an even rank, but -1 of it be of an odd rank.*

205. From the preceding property, it follows, that the two terms of any reduction whatever $\frac{R}{R'}$ are prime to each other.

Indeed let it be supposed, that R and R' have a common factor h; by the preceding property, we have

$$RQ'-R'Q=\pm 1$$

whence dividing both terms by h, we have

$$\frac{RQ'}{h}-\frac{QR'}{h}=\frac{1}{h}$$

but the first member of this equation is an entire number since by hypothesis R and R' are divisible by h, while the second is essentially a fraction; R and R' cannot therefore have a common factor.

From this it follows, that if a fraction, the terms of which are not prime to each other, be converted into a continued fraction, and all the reductions be formed to the last inclusive, the last reduction will not be the proposed fraction, but this fraction reduced to its lowest terms.

Let there be, for example, the fraction $\frac{348}{954}$; converting this into a continued fraction, we have

$$\frac{348}{924}=0+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9}}}}}$$

the reductions of which are $\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{29}{77}$. The last reduction $\frac{29}{77}$ is the proposed reduced to its lowest terms.

206. Since the value of the whole continued fraction x is always comprised between any two consecutive reductions $\frac{Q}{Q'}$, $\frac{R}{R'}$ it follows, that the error committed, in taking $\frac{Q}{Q'}$ for x is less than $\frac{Q}{Q'}-\frac{R}{R'}$; but from what has been demonstrated, we have

$$\frac{Q}{Q'}-\frac{R}{R'}=\frac{1}{Q'R'}$$

and since $Q'<R'$ gives $Q'^2<Q'R'$, we have

$$\frac{1}{Q'R'}<\frac{1}{Q'^2}$$

The error therefore committed in taking any reduction whatever for the value of the whole continued fraction *is less than unity divided by the denominator of this reduction multiplied by the denominator of the reduction which follows*, or less exactly but in terms more simple, *less than unity divided by the square of the denominator of the reduction, which is taken for the whole continued fraction.*

207. The ratio of the circumference to the diameter of a circle being expressed by the fraction $\frac{314159}{100000}$, the terms of

which are prime to each other, let it be proposed to find a fraction, the terms of which will be more simple, and which will express the same ratio nearly. Converting the proposed into a continued fraction, we have

$$\frac{314159}{100000}=3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{25+\cfrac{1}{1+\cfrac{1}{7+\frac{1}{4}}}}}}}$$

which gives for the successive reductions

$$\frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{9208}{2931},\ \frac{9563}{3044},\ \frac{76149}{24239},\ \frac{314159}{100000}$$

The error committed in taking the second of these reductions for the proposed fraction will not exceed $\frac{1}{742}$; $\frac{22}{7}$ is therefore frequently employed to express the ratio of the circumference of a circle to its diameter. This is the ratio given by *Archimedes.*

If a greater degree of approximation is required, we take the fourth reduction, which it is easy to see, is but little more complicated than the third. The error committed in taking this reduction for the proposed will not exceed $\frac{1}{113 \times 2931}$; $\frac{355}{113}$ will therefore approximate the proposed very nearly. This is the ratio given by *Adrian Metius.*

We thus see the use, which may be made of continued fractions in estimating approximatively the value of fractions, the terms of which are large numbers and prime to each other.

SECTION VII.

EXPONENTIAL EQUATIONS AND LOGARITHMS.

208. An equation of the form $a^x = b$, in which the exponent x is the unknown quantity is called an *exponential equation.*

The solution of this equation consists in finding the power, to which it is necessary to raise a given quantity a in order to produce another given quantity b.

Let there be, for example, the equation $2^x = 64$; raising 2 to its different powers, we find soon that $2^6 = 64$; $x = 6$ answers therefore the conditions of the equation.

Again, let there be the equation $3^x = 243$; raising 3 to its different powers, we find $3^5 = 243$; whence $x = 5$. In a word, so long as the second member b is a perfect power of the given number a, x will be an entire number, and its value may be found by raising a successively to its different powers, beginning with that, the exponent of which is 0.

Let it be proposed next to resolve the equation $2^x = 6$. Putting successively $x = 2$, $x = 3$, we have $2^2 = 4$, $2^3 = 8$; the value of x is therefore comprised between the numbers 2 and 3.

Let us put therefore $x = 2 + \frac{1}{x'}$, x' being greater than 1; substituting this value in the proposed, we have

$$2^{2+\frac{1}{x'}} = 6 \text{ or } 2^2 \times 2^{\frac{1}{x'}} = 6, \text{ whence } 2^{\frac{1}{x'}} = \frac{3}{2}$$

or raising both members to the power x', we have

$$\left(\frac{3}{2}\right)^{x'} = 2$$

To determine the value of x', we make successively $x' = 1$, $x' = 2$; thus we have $\left(\frac{3}{2}\right)^1$ or $\frac{3}{2}$ less than 2, but $\left(\frac{3}{2}\right)^2$ or $\frac{9}{4}$ greater than 2; x' is therefore comprised between 1 and 2.

Let us put then $x' = 1 + \frac{1}{x''}$, x'' being greater than 1. Substituting this value, we have

$$\left(\frac{3}{2}\right)^{1+\frac{1}{x''}} = 2 \text{ or } \frac{3}{2} \times \left(\frac{3}{2}\right)^{\frac{1}{x''}} = 2$$

whence $$\left(\frac{4}{3}\right)^{x''} = \frac{3}{2}$$

To determine the value of x'', we make successively

$x'' = 1$, $x'' = 2$; thus we have $\left(\frac{4}{3}\right)^1$ or $\frac{4}{3}$ less than $\frac{3}{2}$, but $\left(\frac{4}{3}\right)^2$ or $\frac{16}{9}$ greater than $\frac{3}{2}$, x'' is therefore comprised between 1 and 2.

Let us put then $x'' = 1 + \frac{1}{x'''}$, x''' being greater than unity; we have by substitution

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}} = \frac{3}{2} \text{ or } \frac{4}{3} \times \left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{3}{2}; \text{ whence } \left(\frac{9}{8}\right)^{x'''} = \frac{4}{3}$$

Making successively $x''' = 1, 2, 3$, we find $\left(\frac{9}{8}\right)^2 = \frac{81}{64}$, a number less than $\frac{4}{3}$ but $\left(\frac{9}{8}\right)^3 = \frac{729}{512}$ a number greater than $\frac{4}{3}$; thus x''' is comprised between 2 and 3.

Let $x''' = 2 + \frac{1}{x''''}$, the equation in x''' becomes

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x''''}} = \frac{4}{3}; \text{ whence } \left(\frac{256}{243}\right)^{x''''} = \frac{9}{8}$$

Operating upon this last equation as upon the preceding, we find two entire numbers k and $k+1$, between which x'''' will be comprised. Putting $x'''' = k + \frac{1}{x^{v}}$, we determine x^{v}, in the same manner as we have already done x'''', and thus in order.

Bringing together the equations

$$x = 2 + \frac{1}{x'},\quad x' = 1 + \frac{1}{x''},\quad x''' = 2 + \frac{1}{x''''} \quad \ldots$$

we obtain the value of x under the form of a continued fraction, thus

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{x''''}}}}}$$

But we have seen that in a continued fraction the greater the number of integrant fractions, which are taken, the nearer we approach the value of the number reduced to a continued fraction; we shall therefore be able to determine the value of x in the equation $2^x = 6$, if not exactly, at least with such degree of approximation as we please.

Forming the first four reductions, for example, we have $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{13}{5}$; and the reduction $\frac{13}{5}$ differs from x by a quantity less than $\frac{1}{25}$.

To attain a greater degree of approximation, we determine the value of x'''' in the equation $\left(\frac{256}{243}\right)^{x''''} = \frac{9}{8}$; we thus find $x'''' = 2 + \frac{1}{x^{\text{v}}}$. We shall have then for the fifth reduction $\frac{31}{12}$. This differs from x by a quantity less than $\frac{1}{144}$.

209. From the preceding examples the course to be pursued in the solution of equations of the form $a^x = b$ will be readily inferred. In the application of this method to particular cases it is necessary to remark, 1°. If the quantity b be less than a, the value of x will be comprised between 0 and 1; we put therefore $x = \frac{1}{x'}$. 2°. If b is a fraction and a greater than unity, the value of x will be negative, we put therefore $x = -y$; the equation is then reduced to the form $a^y = \frac{1}{b}$; having found the value of y in this equation according to the method explained above, the value of x will be equal to that of y taken *negatively*.

EXAMPLES.

1°. Given $3^x = 15$ to find the value of x. Ans. $x = 2.46$

2°. . . $10^x = 3$ . . . Ans. $x = 0.47$

3°. Given $5^x = \frac{2}{3}$ to find the value of x. Ans. $x = -0.25$

4°. . . $\left(\frac{7}{12}\right)^x = \frac{3}{4}$ . . . . Ans. $x = 0.53$

In the above examples the reductions furnished by the method are converted into decimal fractions, and the value of x is determined to within .01.

THEORY OF LOGARITHMS.

210. If in the equation $a^x = y$, we assign a constant value different from unity to a, and suppose that of x to vary, as may be required, we may obtain successively for y all possible numbers.

Let us suppose first a greater than 1.

If we make successively $x = 0,\ 1,\ 2,\ 3,\ 4, \ldots$

we have $y = 1,\ a,\ a^2,\ a^3,\ a^4 \ldots$

Thus by means of the powers of a, the exponents of which are positive entire or fractional, we may produce all possible positive numbers greater than 1.

Again let $x = 0, -1, -2, -3, -4 \ldots$

we have $y = 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4} \ldots$

Thus by means of the powers of a, the exponents of which are negative entire or fractional, we may produce all possible positive numbers less than 1.

If on the other hand we suppose a less than unity, still all possible positive numbers may be produced by means of the different powers of a, only the order in which they are produced will be reversed.

We see therefore, that *all possible positive numbers may be produced by means of any positive number whatever* a, *different from unity, by raising this number to the requisite powers*.

It is necessary, that a should be different from unity, otherwise the same number will be produced, whatever value we assign to x.

211. Let it now be supposed that we have made a table

containing in one column all entire numbers, and by the side of these in another column the exponents of the powers, to which it is necessary to raise a constant number in order to produce these numbers ; this would be *a table of logarithms*.

We define the logarithm of a number therefore, *the exponent of the power, to which it is necessary to raise a given or invariable number, in order to produce the proposed number.*

Thus in the equation $a^x = y$, x is the logarithm of y ; in like manner in the equation $2^6 = 64$, 6 is the logarithm of 64. The logarithm of a number is indicated by writing before it the first three letters of the word logarithm, or more simply by placing before it the letter L.

The invariable number, from which the others are formed, is called the *base* of the table. It may be taken at pleasure either greater or less than unity, but should remain the same for the formation of all numbers, that belong to the same table.

Since $a^0 = 1$, and $a^1 = a$, whatever number may be assumed for the base of the table, *the logarithm of the base will be unity, and the logarithm of unity will be* 0.

212. We proceed to show the properties of logarithms in relation to numerical calculations.

Let there be the series of numbers $y, y', y'' \ldots$ to be multiplied together. Let a represent the base of a system of logarithms, which we suppose already calculated, and let $x, x', x'' \ldots$ be the logarithms of $y, y', y'' \ldots$; by the definition of a logarithm we have

$$y = a^x,\ y' = a^{x'},\ y'' = a^{x''}$$

multiplying these equations member by member, we have

$$y\,y'\,y'' = a^{x+x'+x''}$$

whence; $\log y\,y'y'' = x + x' + x'' = \log y + \log y' + \log y''$

That is, *the logarithm of a product is equal to the sum of the logarithms of the factors of this product.*

If then a multiplication be proposed, we take from a table of logarithms the logarithms of the numbers to be mul-

tiplied; the sum of these logarithms will be the logarithm of the product sought. Seeking therefore this logarithm in the table, the number corresponding to it will be the product sought. Thus by means of a table of logarithms *addition may be made to take the place of multiplication.*

Again let it be required to divide the number y by the number y'; let x, x' be the logarithms of these numbers, we have the equations

$$y = a^x, \quad y' = a^{x'}$$

dividing these equations member by member, we have

$$\frac{y}{y'} = \frac{a^x}{a^{x'}} = a^{x-x'}$$

whence $\quad \log \frac{y}{y'} = x - x' = \log y - \log y'$

That is, *the logarithm of a quotient is equal to the difference between the logarithm of the divisor and that of the dividend.*

If then it be proposed to divide one number by another, from the logarithm of the dividend we subtract the logarithm of the divisor, the result will be the logarithm of the quotient; seeking therefore this logarithm in the tables the number corresponding will be the quotient sought. Thus by means of a table of logarithms *subtraction may be made to take the place of division.*

Let it next be required to raise the number y to the power denoted by m, we have the equation

$$a^x = y$$

raising both members to the mth power, we have

$$a^{mx} = y^m$$

whence the logarithm of $y^m = m\,x = m \log y$.

That is, *the logarithm of any power of a number is equal to the product of the logarithm of this number by the exponent of the power.*

To form any power whatever of a number by means of a table of logarithms, we multiply therefore the logarithm of the proposed number by the exponent of the power, to

which it is to be raised; the number in the table corresponding to this product, will be the power sought.

Again let it be required to find the nth root of y. We have as before

$$a^x = y$$

whence taking the nth root of both members, we have

$$a^{\frac{x}{n}} = y^{\frac{1}{n}}$$

whence $\quad \log y^{\frac{1}{n}} = \frac{x}{n} = \frac{\log y}{n}$

That is, *the logarithm of the root of any degree whatever of a number is equal to the logarithm of this number divided by the index of the root.*

Thus by the aid of a table of logarithms *a simple multiplication may be made to take the place of the formation of a power, and a simple division that of the extraction of a root.*

FORMATION OF TABLES.

213. The properties of logarithms demonstrated above are altogether independent of the number a or their base. We may therefore form an infinite variety of tables of logarithms by putting for a all possible numbers except unity. The most convenient number, however, for a base and the one employed in the construction of the tables in common use is 10.

214. To calculate a table of logarithms the base of which is 10, we make, in the equation $10^x = y$, y equal successively to the numbers 1, 2, 3 . . ., and determine by the methods explained art. 209 the values of x corresponding. We thus obtain the values of x exactly, if y be a perfect power of 10, or otherwise with such degree of approximation as we please.

To calculate the logarithm of 1000, for example, we have the equation $10^x = 1000$, from which we obtain $x = 3$; thus the logarithm of 1000 is 3.

To calculate the logarithm of 2, we have the equation $10^x = 2$, from which we deduce

$$x = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{9 + \cfrac{1}{x'''}}}}$$

Whence stopping at the third integrant fraction, and forming the reduction corresponding, we have $x = \frac{28}{93}$ or reducing this last to a decimal, we have $x = .30107$; the logarithm of 2 is therefore .30107, accurate to the fourth decimal figure.

215. In the calculation of a table of logarithms, it will be sufficient to calculate directly the logarithms of the prime numbers 1, 2, 3, 5 . . ., the logarithms of compound numbers may then be obtained by adding the logarithms of the prime factors, which enter into them. To find the logarithm of 35, for example, we have $35 = 5 \times 7$; whence $\log 35 = \log 5 + \log 7$; having already calculated the logarithms of 5 and 7, the logarithm of 37 will be found therefore by adding the logarithm of 5 to that of 7.

Since moreover the logarithm of a fraction will be equal to the logarithm of the numerator minus the logarithm of the denominator, it will be sufficient to place in the tables the logarithms of entire numbers.

216. If in the equation $10^x = y$ we make successively

$$x = 0,\ 1,\ 2,\ 3,\ 4\ \ldots$$

we have

$$y = 1,\ 10,\ 100,\ 1000,\ 10000\ \ldots$$

Again if we make

$$x = 0,\ -1,\ -2,\ -3,\ -4\ \ldots$$

we have

$$y = 1,\ \frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000},\ \frac{1}{10000}\ \ldots$$

Therefore in a table of logarithms, the base of which is 10, 1°. the logarithms of numbers greater than unity are positive and go on increasing from 0 to infinity. 2°. the logarithms of numbers less than unity are negative, and their ab-

solute values are so much the greater as the fractions are smaller; whence if we take a fraction less than any assignable quantity, the logarithm of this fraction will be negative, and its absolute value will be greater than any assignable quantity. On this account we say that the logarithm of 0 is an *infinite negative quantity*. 3°. The logarithms of all numbers below 10 are fractions; the logarithms of numbers between 10 and 100 are 1 and a fraction; the logarithms of numbers between 100 and 1000 are 2 and a fraction; those of numbers between 1000 and 10000 are 3 and a fraction; and in general, the whole number, which precedes the fraction in the logarithm is less by one than the number of figures in the number corresponding to the logarithm. On this account it is called the *index or characteristic* of the logarithm, since it serves to indicate the order of units, to which the number corresponding to the logarithm belongs. Thus in the logarithm 3.75527 the characteristic 3 shows that the number corresponding to this logarithm consists of 4 figures or is comprised between 1000, and 10000.

217. The logarithm of a number being given, the logarithm of a number 10, 100, . . . times greater is found by adding 1, 2, . . . units to the characteristic only; indeed $\log\ (y \times 10^n) = \log y + \log 10^n = \log y + n$; whence it will be sufficient to add n units to the logarithm of y in order to obtain the logarithm of a number 10^n times as great, an addition, which may be performed upon the characteristic only. Conversely, $\log \frac{y}{10^n} = \log y - \log 10^n = \log y - n$; thus it is sufficient to subtract n units from the logarithm of y, in order to find the logarithm of a number 10^n times smaller than y.

218. The fractional parts of logarithms in the tables are expressed by decimals. From what has been said the decimal part of the logarithm of a number will be the same for this number multiplied or divided by 10, 100, . . . On this account the system of logarithms, the base of which is 10,

is more convenient than any other system, since we have frequent occasion to multiply or divide by 10, 100, . . . operations reduced in this case to the simple addition or subtraction of units.

219. Since the characteristic of the logarithm may be easily determined by the number, and the number of figures in the number by the characteristic of the logarithm, it is usual to omit the characteristic in the tables to save the room. It is also convenient to omit it; because the same decimal part with different characteristics forms the logarithms of several different numbers.

220. Having already calculated a system of logarithms, it will be easy from this to form as many other systems as we please.

Indeed, let N designate any number whatever, log N its logarithm in the system the base of which is a, X its logarithm in a different system, the base of which is b, we have

$$b^{X} = N$$

taking the logarithms of both members of this equation in the system, the base of which is a, we have

$$X . \log b = \log N$$

whence
$$X = \frac{\log N}{\log b}$$

Having calculated therefore a set of tables for a particular base, to find the logarithm of a number in a proposed system with a different base, *we take from the tables already calculated the logarithm of the number, and also the logarithm of the base of the proposed system, the former of these logarithms, divided by the latter, will give the logarithm of the number in the proposed system.*

The logarithm of 6, for example, in the system, the base of which is 10, is .77815 and that of 2 is .30103; the logarithm of 6 therefore in the system the base of which is 2 will be

$$\frac{.77815}{.30103} = 2.58495$$

221. The expression $X = \frac{\log N}{\log b}$ may be put under the form $X = \log N. \frac{1}{\log b}$. Thus having already formed a system of logarithms the base of which is a, to construct from this a new system, the base of which shall be b, we multiply the logarithms of the first system by the quantity $\frac{1}{\log b}$. This quantity, by means of which we are enabled to pass from the first to the second system, is called the *modulus* of the second system in relation to the first.

USE OF THE TABLES.

222. As it is impossible to place in the tables the logarithms of all numbers, it is usual to place in them the logarithms of numbers from unity to within a certain limit. In what follows, it is supposed, that the student has in his hands tables containing the logarithms of entire numbers from 1 to 10000.

In order to use such a set of tables, we have the two following questions to resolve, viz. 1°. *any number whatever being given, to find its logarithm.* 2°. *any logarithm being given, to find the number which corresponds to it.*

The following examples will exhibit the method of resolving these questions.

1. Let it be proposed to find the logarithm of 9748. Seeking the proposed in the column of numbers against it in the column of logarithms we find 98892; this will be the decimal part of the logarithm; or as is the case with most tables, if the column of numbers contain but three places of figures, we look for 974 the first three figures of the proposed in the first column, and at the top of the table we look for the fourth figure 8; directly under the 8 and in the same line with 974, we find the decimal part 98892 as before; then since the proposed consists of four places the characteristic will be 3, thus $\log 9748 = 3.98892$.

2. Let it be required to find the logarithm of 76.93.

Removing for the moment the decimal point, we find as above log 7693 = 3.88610, whence, art. 217, subtracting 2 units from the characteristic 3 of this logarithm, we shall have the logarithm of the proposed; thus log 76.93 = 1.88610.

3. To find the logarithm of .75. The logarithm of this number may be presented under two different forms. Writing it in the form of a vulgar fraction, it becomes $\frac{75}{100}$: The logarithm of 75 is 1.87506, and that of 100, is 2.00000; whence subtracting the logarithm of the denominator from that of the numerator, art. 212, we have — 12494 = log .75. This logarithm being altogether negative is inconvenient in practice; it will be observed, however that $.75 = \frac{1}{100} \times 75$;

$$\text{whence } \log \frac{1}{100} + \log .75 = -2 + 1.87506,$$
$$= -1 + 87506$$

or placing the sign — over the 1 to show that the characteristic only is negative, we have $\log .75 = \bar{1}\ 87506$.

This last form of the logarithm of the proposed is derived, it will be perceived, immediately from the continuation of the principle art. 219, according to which the logarithm of a number 10, 100 . . . times less than a proposed number is found by subtracting 1, 2 . . . units from the logarithm of its characteristic.

Thus the logarithm of 75 being 1. 87506, we have

$$\log 7.5 = 0.\ 87506$$
$$\log .75 = \bar{1}.87506$$
$$\log .075 = \bar{2}.\ 87506$$
$$\cdots\cdots\cdots$$

4. To find the logarithm of $\frac{4}{5}$; we have log 4 = .60206, log 5 = .69897; whence subtracting this last logarithm from the former, we have $\log \frac{4}{5} = -.09691$, in which the logarithm is entirely negative. But $\frac{4}{5}$ reduced to a decimal becomes .8, the logarithm of which is $\bar{1}.90309$, the characteristic only being negative.

5. To find the logarithm of $54\frac{7}{9}$; we have $54\frac{7}{9}=\frac{493}{9}$; log .493 $=2.69285$, log $9=0.95424$; whence subtracting the latter logarithm from the former, we have log. $\frac{493}{9}$ or $54\frac{7}{9}=1.73861$.

6. To find the logarithm of 675437. This number exceeds the limits of the table; its logarithm, however, may be readily found. The greatest number of places in a number, the logarithm of which can be found in the tables is 4; separating therefore the four left hand figures of the proposed from the rest by a point, we consider, for the moment, those on the left as decimals. The logarithm of 6754.37 is comprised between the logarithm of 6754 and that of 6755; the difference between these two logarithms is .00007; $\frac{37}{100}$ of this difference therefore added to the less logarithm will give the logarithm of 6754.37 nearly; thus log 6754.37 $=$ 3.82959; whence adding 2 units to the characteristic of of this last to obtain the logarithm of the proposed, we have log 675437 $=5.82959$.

223. We proceed next to the second of the proposed questions, viz. *A logarithm being given to find the number, which corresponds to it.*

1. To find the number corresponding to the logarithm 2.10449. The decimal part of this logarithm is contained in the tables; in the left hand column and on the same line with it, according to the arrangement of the tables, in which there are but three places of figures in the column of numbers, we find 127, and at the top of the table directly over it we find 2; the characteristic of the logarithm being 2, we have therefore 127.2 for the number corresponding to the proposed.

2. To find the number corresponding to the logarithm 3.42674; this logarithm is not found in the tables; it is comprised however between 3.42667 the logarithm of 2671, and 3.42684 that of 2672; the difference between these two logarithms is .00017, the difference between the proposed

and 3. 42667 is .00007 ; we have then the following proportion

.00017 : 1 : : 00007 : .41 nearly

The number corresponding to 3.42674 is therefore 2671.41.

3. To find the number corresponding to the logarithm — 2. 45379. The number corresponding to this logarithm will be comprised, it is evident, between .01 and .001 ; to obtain this number let us add to — 2. 45379 a sufficient number of units to make it positive, 5 for example, we have 5 — 2. 45379 = 2. 54621 ; the number corresponding to this last is 351.73 ; but by adding 5 units to the proposed logarithm, we have multiplied the number, to which it belongs by 100 000, whence dividing 351.73 by 100 000, we have .0035173, the number corresponding to the proposed.

4. To find the number corresponding to the logarithm $\bar{3}$ .86249. Adding 3 units to the characteristic, the proposed becomes 0. 86249, the number corresponding to which is 72.86 ; whence, as it is easy to see, the number corresponding to $\bar{3}$.86249 is .007286.

APPLICATION OF THE THEORY OF LOGARITHMS.

MULTIPLICATION AND DIVISION.

1. Let it be required to multiply 762 by .097.

$$\begin{array}{lr} & \log 872 = 2.94052 \\ & \log .097 = \bar{2}.98677 \\ \hline \log 84.584 \textit{ Ans.} & 1.92729 \end{array}$$

2. Let it be required to multiply .857 by .0093.

$$\begin{array}{lr} & \log .857 = \bar{1}.93298 \\ & \log .0093 = \bar{3}.96848 \\ \hline \log .00797 \textit{ Ans.} & \bar{3}.90146 \end{array}$$

3. Let it be required to divide 5672 by .0037.

$$\begin{array}{lr} & \log 5672 = 3.75374 \\ & \log .0037 = \bar{3}.56820 \\ \hline \log 1533000 \textit{ Ans.} & 6.18554 \end{array}$$

4. Let it be required to divide .053 by 797.

$$\begin{array}{ll} \log .053 = \bar{2}.72428 = \bar{3} + & 1.72428 \\ \log\ 797 = & \underline{2.90146} \\ \log .0000665\ \textit{Ans.} & \bar{5}.82282 \end{array}$$

To render the subtraction required in this example possible, we change the characteristic $\bar{2}$ into $\bar{3}+1$, which has the same value; this furnishes a ten to be joined with 7 for the subtraction of 9, the last figure of the positive part A similar preparation, it is evident, must be made in all cases of the same kind.

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

124. 1. Let it be required to find the 5th power of .125.

$$\begin{array}{rr} \log .125 = & \bar{1}.09691 \\ & \underline{5} \\ \log .000030519\ \textit{Ans.}\ \text{nearly} & \bar{5}.48455 \end{array}$$

2. To find the 7th power of .73

$$\begin{array}{rr} \log .73 = & \bar{1}.86332 \\ & \underline{7} \\ \log .11022\ \textit{Ans.}\ \text{nearly} & \bar{7} + 6.04224 = \bar{1}.04224 \end{array}$$

3. To find the third root of .01356.

The logarithm of .01356 is $\bar{2}$. 13226. The negative characteristic $\bar{2}$ of this logarithm is not divisible by 3, the index of the root required, neither can it be joined to the positive part on account of the different sign. If however we add $-1+1$ to the characteristic, which will not alter its value, it becomes $\bar{3}+1$; the negative part is then divisible by 3, and the 1 being positive may be joined to the fractional part, we have then

$$\log .01356 = \bar{2}.13226 = \bar{3} + 1.13226$$

whence dividing by 3, we have

$$\bar{1}.37742 = \log\ .23845\ \textit{Ans.}\ \text{nearly}.$$

In all cases, if the negative characteristic is not divisible by the index of the root required, it must be made so in a similar manner.

ARITHMETICAL COMPLEMENT.

225. The arithmetical complement of a logarithm is the difference between this logarithm and 10; thus the arithmetical complement of 3.472584 is $10 - 3.472584 = 6.527416$. The arithmetical complement of a logarithm is obtained *by subtracting the right hand figure, if it be significant, from* 10, *and the others from* 9.

Let it be proposed to find the value of x in the expression

$$x = l - l' + l'' - l''' - l''''$$

l, l', l'' . . being logarithms; this expression, it is evident, may be put under the form

$$x = l + (10 - l') + l'' + (10 - l''') + (10 - l'''') - 30$$

that is, to find the value of x, *we take the sum of the logarithms to be added and the complements of the logarithms to be subtracted, and from this sum subtract as many times* 10, *as there are complements employed.*

Thus when there are several multiplications and divisions to be performed together, by using the complements of the logarithms of the divisors the whole may be reduced to the addition of logarithms.

Let it be proposed, for example, to find the value of x in the expression

$$x = \frac{31}{95} \times \frac{139}{457}$$

log 31 =	1,49136
log 139 =	2.14301
log 95 Comp. =	8.02228
log 457 Comp. =	7.34008
	18.99673
Subtract	20
log .09925	$\bar{2}$.99673

Let it be proposed, as a second example, to find the value of x in the expression

$$x=\left(\frac{3.75\times 73\times .056}{1.7498\times 125.13}\right)^{\frac{5}{3}}$$

log 3.75	0.57403
log 73	1.86332
log .056	$\bar{2}.74819$
log 1.7498 Comp.	9.75701
log 125.13 Comp.	7.90264
	18.84519
Subtract	20
	$\bar{2}.84519$
	5
product by 5	$\bar{6}.22595$ (3
quotient by 3	$\bar{2}.07532=$ log .011893 *Ans.*

PROPORTIONS.

Let it be required to find the fourth term of the proportion, of which the numbers 963, 1279, 8.7 are the first three terms.

log 1279	3.10687
log 8.7	0.93952
log 963 Comp.	7.01637
log 11.528 *Ans.* nearly	1.06276

From the proportion $a : b :: c : d$, we have

$$\frac{a}{b}=\frac{c}{d}$$

whence $\quad \log a-\log b=\log c-\log d$

therefore $\quad \log a \,.\, \log b : \log c \,.\, \log d$

that is, *if four numbers form a proportion, their logarithms will form an equidifference.*

EXPONENTIAL EQUATIONS.

226. We have already explained a method for finding the value of x in the equation $a^x = b$, from which the theory of logarithms is derived; but a table of logarithms being once constructed, there is nothing to prevent its use in the solution of equations of this kind.

Let it be required to find the value of x in the equation $3^x = 15$.
taking the logarithms of both sides, we have

$$x \log 3 = \log 15$$

whence
$$x = \frac{\log 15}{\log 3} = \frac{1.17609}{.47712} = 2.464 +$$

The division required in this example may be performed, it is easy to see, by subtracting the logarithm of .47712 from that of 1.17609, as in the case of any other numbers.

PROGRESSIONS BY QUOTIENT.

227. Logarithms are particularly useful in the solution of questions in progression by quotient.

Let it be proposed to find the 20th term in the progression $1 : \frac{3}{2} : \frac{9}{4} : \frac{27}{8} \ldots$

Putting u for the last term of a progression by quotient, we have (art. 189),

$$u = a q^{n-1}; \text{ whence } \log u = \log a + (n-1) \log q$$

We have therefore for the 20th term in the progression proposed

$$\log u = \log 1 + 19 (\log 3 - \log 2) = 19 (\log 3 - \log 2)$$

the term required will therefore be 2216.84 to within .01.

Let it be required next to insert between the numbers 2 and 15 fifty mean proportionals; we have for the ratio, art. 198.

$$q = \sqrt[m+1]{\frac{b}{a}}; \text{ whence } \log q = \frac{\log b - \log a}{m+1},$$

in the question proposed, we have therefore

$$q = \frac{\log 15 - \log 2}{51}$$

or performing the calculations, we obtain

$$q = 1.040299$$

228. Let it be required to find the sum of the first ten terms in the progression $\div\!\div 5.15.45 \ldots$; we have, art. *192*,

$$S = \frac{a(q^n - 1)}{q-1};\ \text{whence}$$

$$\log S = \log a + \log (q^n - 1) - \log (q - 1)$$

Applying this formula to the proposed question, we have

$$\log S = \log 5 + \log (3^{10} - 1) - \log (3 - 1)$$

Calculating 3^{10} by logarithms, we have

$$\log 3^{10} = 10 \times \log 3$$

from which we obtain $3^{10} = 59048$

whence $\log S = \log 5 + \log 59047 - \log 2$

or performing the calculations, we obtain 147620 for the sum required.

Let it be proposed next to find the number of terms in the progression, of which the first term is 3, the ratio 2, and the last term 6144.

From the formula $u = a\,q^{n-1}$ we have

$$\log u = \log a + (n - 1) \log q\,;$$

whence $$n = 1 + \frac{\log u - \log a}{\log q}$$

Applying this formula to the proposed question, we have

$$n = 1 + \frac{\log 6144 - \log 3}{\log 2} = 1 + 11 = 12$$

229. Let us take next the progression

$$\div\!\div a : b : c : d : e : f : g \ldots$$

from the nature of the progression, we have

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{f}{g} \ldots$$

whence $\log \frac{a}{b} = \log \frac{b}{c} = \log \frac{c}{d} = \log \frac{d}{e} \ldots$

wherefore, $\log a - \log b = \log b - \log c = \log c - \log d = \ldots$
but from this last we have

$$\div \log a \,.\, \log b \,.\, \log c \,.\, \log d \ldots$$

If therefore the numbers a, b, c, d . . form a progression by quotient, their logarithms will form a progression by difference. Logarithms may therefore be defined *a series of numbers in arithmetical progression corresponding term to term to another series of numbers in geometrical progression.* This is the definition of logarithms given in arithmetic.

COMPOUND INTEREST.

230. One of the most important applications of logarithms is to questions upon the interest of money.

Interest is of two kinds *simple* and *compound.* If interest be paid upon the principal only, it is called *simple interest*; but if the interest as it becomes due be added to the principal, and interest be paid upon the whole, it is then called *compound interest.*

We have already investigated formulas for simple interest. Let it now be proposed to determine what sum a given principal p will amount to, in a number n of years, at a given rate r per cent. at compound interest.

The amount of $1 for one year will be $1 + r$; that of p dollars will be therefore $p(1 + r)$.

For the second year $p(1 + r)$ will be the principal, and its amount will be $p(1 + r)(1 + r)$ or $p(1 + r)^2$.

The original sum p therefore at the end of the second year will amount to $p(1 + r)^2$. In like manner at the end of the third year it will amount to $p(1 + r)^3$; whence putting A for the amount required, we have

$$A = p(1 + r)^n$$

this is a general formula for compound interest; taking the logarithms of both sides, we have

$$\log A = \log p + n \log (1 + r)$$

Let it be proposed to determine what sum \$30000 will amount to, in 30 years, at 5 per cent. compound interest.

We have $\log A = \log 30000 + 30 \log 1.05$

whence performing the calculations, we obtain

\$129658.27, *Ans.*

231. The equation $A = p(1+r)^n$ contains four quantities A, p, r, and n, any one of which may be determined, when the others are known. It gives rise therefore to the four following questions, viz.

1°. *To determine* A, *when* p, r *and* n *are given, or the principal, rate, and number of years being given, to find the amount.*

This question we have already solved.

2°. *To determine* p, *when* A, r *and* n *are given,* or *to find what principal put at compound interest will amount to a given sum, in a certain number of years, at a given rate per cent.*

Resolving the general equation with reference to p, we have

$$p = \frac{A}{(1+r)^n}$$

or by logarithms $\log p = \log A - n \log (1+r)$

3°. *To determine* r, *when* A, p *and* n *are known,* that is, *to find at what rate a given sum must be put at compound interest, in order to amount to another given sum in a given time.*

Resolving the general equation with reference to r, we have

$$(1+r) = \sqrt[n]{\frac{A}{p}}$$

or by logarithms $\log (1+r) = \frac{\log A - \log p}{n}$

Having by means of this last determined the value of $1+r$, that of r will be easily found.

4°. *To determine* n, *when* A, p *and* r *are given,* that is, *to find for what time a given sum must be put at compound interest, at a certain rate per cent, in order to amount to a given sum.*

Making n the unknown quantity in the general formula, we obtain

$$n = \frac{\log A - \log p}{\log (1+r)}$$

If it be asked what must be the value of n in order that the sum at interest may be doubled, tripled, &c ; we put in the general formula $A = kp$, k denoting $1, 2, 3 \ldots$, we thus have

$$kp = p(1+r)^n ; \text{ whence } n = \frac{\log k}{\log (1+r)}$$

n is therefore independent of p, that is, whatever the sum put out, it will be doubled, tripled, &c. in the same time.

EXAMPLES.

1. What is the amount of $1000 for 25 years at 5 per cent compound interest? Ans. $3386.

2. What is the amount of £217 for $2\frac{1}{4}$ years, at 5 per cent, supposing the interest payable quarterly.
Ans. £242. 13s. $4\frac{1}{2}$ d.

3. A note was given the 4th of June, 1809, for $75.93 at the rate of 6 per cent compound interest; and it was paid the 21st of August 1825. To what had it amounted?

4. What principal put at interest will amount to $16081 in 6 years at 5 per cent?

5. What principal put at interest will amount to $30000 in 7 years at 5 per cent?

6. At what rate per cent will a principal of $3452 amount to $143763 at the end of 64 years.

7. In what time will $435.40 amount to 735.66\frac{3}{4}$ at 6 per cent?

8. In what time will a principal p be doubled at 5 per cent? In what time will it be tripled at 6 per cent?

ANNUITIES.

232. An annuity is a sum of money payable yearly for a certain number of years or forever.

Let it be proposed to determine what sum must be put at interest to pay an annuity of b dollars for n years, the interest being reckoned at r per cent compound interest.

According to the rule for compound interest, the amount of the first payment, at the expiration of the n years, will be $b\,(1+r)^{n-1}$, the amount of the second payment will be $b\,(1+r)^{n-2}$, that of the third will be $b\,(1+r)^{n-3}$ . . . . . the last payment will be b. Putting A for the sum placed at interest for the payment of the annuity, its amount at the end of the n years will be $A\,(1+r)^n$; we shall have therefore

$$A\,(1+r)^n = b\,(1+r)^{n-1} + b\,(1+r)^{n-2} + b\,(1+r)^{n-3} \ldots b$$

but the second member of this equation forms, it is evident, a progression by quotient; taking its sum, we have

$$A\,(1+r)^n = \frac{b\,[\,(1+r)^n - 1]}{r}$$

whence
$$A = \frac{b\,[\,(1+r)^n - 1]}{r\,(1+r)^n}.$$

This equation gives rise also to four different questions, according as we make A, b, r or n the unknown quantity. The following examples exhibit particular cases of these questions.

1. A man wishes to purchase an annuity, which shall afford him \$1500 a year for 12 years; What sum must he deposit in the annuity office to produce this sum, supposing he can be allowed $7\frac{1}{2}$ per cent interest?

2. A man purchased an annuity for 15 years for \$100000. How much can he draw annually, the interest being reckoned at 5 per cent?

3. A man has property to the amount of \$40000, which yields him an income of 5 per cent. His annual expenses are \$6000. How long will his property last him?

Section VIII.

BINOMIAL THEOREM. INDETERMINATE COEFFICIENTS.

233. The rule given art. 147 for the development of $(x+a)^m$, for the case in which m is entire and positive, is equally applicable whatever may be the value of the exponent m. Indeed if we divide unity by each member of the formula

$$(x+a)^m = x^m + m\,a\,x^{m-1} + \frac{m\,(m-1)}{1.2}\,a^2\,x^{m-2} + \ldots\ldots$$

the quotients, it is evident, should be equal; we have therefore

$$(x+a)^{-m} = x^{-m} - m\,a\,x^{-m-1} + \frac{-m\,(-m-1)}{1.2}\,a^2 x^{-m-2} + \ldots$$

extracting next the nth root of the above formulas, we have

$$(x+a)^{\frac{m}{n}} = x^{\frac{m}{n}} + \frac{m}{n}\,a\,x^{\frac{m}{n}-1} + \frac{1}{2}\,\frac{m}{n}\left(\frac{m}{n}-1\right)a^2\,x^{\frac{m}{n}-2} + \ldots\ldots$$

$$(x+a)^{-\frac{m}{n}} = x^{-\frac{m}{n}} - \frac{m}{n}\,a\,x^{-\frac{m}{n}-1} + \frac{1}{2}\left(-\frac{m}{n}\right)\left(-\frac{m}{n}-1\right)a^2 x^{-\frac{m}{n}-2}$$

Comparing the different terms of the second, third and fourth of the developments above with the corresponding terms of the first, it will be perceived, that they are formed according to the same law.

EXAMPLES.

1. Let it be proposed to find the development of $(x+a)^{\frac{2}{3}}$; we have by the rule

$$(x+a)^{\frac{2}{3}} = x^{\frac{2}{3}} + \frac{2}{3}\,x^{-\frac{1}{3}}\,a - \frac{2.1}{3.6}\,x^{-\frac{4}{3}}\,a^2 + \frac{2.1.4}{3.6.9}\,x^{-\frac{7}{3}}\,a^3 \ldots\ldots$$

or reducing

$$(x+a)^{\frac{2}{3}} = x^{\frac{2}{3}} + \frac{2\,x^{\frac{2}{3}}}{3\,x}\,a - \frac{2.1\;x^{\frac{2}{3}}}{3.6\;x^2}\,a^2 + \frac{2.1.4\;x^{\frac{2}{3}}}{3.6.9\;x^3}\,a^3 \ldots\ldots$$

2. Let it be proposed next to find the development of $(x+a)^{-3}$. Applying the rule, we have

$$(x+a)^{-3}=x^{-3}-3x^{-4}a+\frac{3.4}{2}x^{-5}a^2-\frac{3.4.5}{2.3}x^{-6}a^3\ \ldots$$

or reducing

$$(x+a)^{-3}=\frac{1}{x^3}-\frac{3a}{x^4}+\frac{6a^2}{x^5}-\frac{10a^3}{x^6}\ \ldots\ .$$

234. The binomial formula furnishes a method of finding the approximate roots of numbers, which are not perfect powers.

If in the expression for $(x+a)^{\frac{m}{n}}$ we put $m=1$, we have

$$(x+a)^{\frac{1}{n}}=x^{\frac{1}{n}}+\frac{1}{n}ax^{\frac{1}{n}-1}+\frac{1}{2}\frac{1}{n}\left(\frac{1}{n}-1\right)a^2x^{\frac{1}{n}-2}\ \ldots\ .$$

or reducing $(x+a)^{\frac{1}{n}}=$

$$x^{\frac{1}{n}}+\frac{1}{n}\cdot\frac{ax^{\frac{1}{n}}}{x}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2x^{\frac{1}{n}}}{x^2}+\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3x^{\frac{1}{n}}}{x^3}-\ldots$$

but this last may be put under the form $(x+a)^{\frac{1}{n}}=$

$$x^{\frac{1}{n}}\left(1+\frac{1}{n}\cdot\frac{a}{x}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2}+\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3}-\ldots\right)\quad(1)$$

To form a new term of this development, it will be sufficient to multiply the fourth by $\frac{3n-1}{4n}$ and by $\frac{a}{x}$ and to change the sign, and thus in order.

Substituting $-a$ for a in the formula (1), we have $(x-a)^{\frac{1}{n}}$

$$=x^{\frac{1}{n}}\left(1-\frac{1}{n}\cdot\frac{a}{x}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3}-\ldots\right)\quad(2)$$

Let it now be proposed to find the approximate square root of 6. In order to this we decompose 6 into two parts $4+2$, the first of which is a perfect square, we have then

$6^{\frac{1}{2}} = (4+2)^{\frac{1}{2}}$; putting therefore in the formula (1) $x=4$, $a=2$, we have

$$6^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1 + \frac{1}{2}\cdot\frac{2}{4} - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{4}{16} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{8}{64}\cdot\ldots\right)$$

or reducing $$6^{\frac{1}{2}} = 2 + \frac{1}{2} - \frac{1}{16} + \frac{1}{64} \ldots$$

To find the next term, we multiply, according to what has been said, $\frac{1}{64}$ by $\frac{3n-1}{4n}\cdot\frac{a}{x}$ or $\frac{5}{8}\cdot\frac{2}{4}$ and change the sign; this term will be therefore $-\frac{5}{1024}$.

If we take the first two terms of this series, we shall have $6^{\frac{1}{2}} = \frac{5}{2}$, the square of which $\frac{25}{4}$ is greater than 6 by the fraction $\frac{1}{4}$. If we take the first three terms, we shall have $6^{\frac{1}{2}} = \frac{39}{16}$, the square of which is less than 6 by the fraction $\frac{15}{256}$.

In order that the series may converge more rapidly, it is necessary, as will be evident from inspection of the general formulas, to make the second term of the binomial into which the proposed is converted as small as possible.

Thus in the above example, if we put $6^{\frac{1}{2}} = \left(\frac{25}{4} - \frac{1}{4}\right)^{\frac{1}{2}}$ and make in formula (2) $x = \frac{25}{4}$, $a = \frac{1}{4}$, we have, the reductions being made,

$$6^{\frac{1}{2}} = \frac{5}{2} - \frac{1}{20} - \frac{1}{2000} - \ldots$$

a series, which converges much more rapidly than the one first obtained. Indeed, if we take the first two terms of this series, we have $6^{\frac{1}{2}} = \frac{49}{20}$, the square of which differs from 6 by $\frac{1}{400}$ only.

Let it be proposed next to find the approximate third root of 7. Putting in formula (1) $x=1$, $a=6$, we have

$$7^{\frac{1}{3}}=1\left(1+\frac{1}{3}\cdot\frac{6}{1}-\frac{1}{3}\cdot\frac{2}{6}\cdot\frac{36}{1}+\frac{1}{3}\cdot\frac{2}{6}\cdot\frac{5}{9}\cdot\frac{216}{1}-\ .\ .\ .\right)$$

a series, in which the terms increase continually instead of diminishing. But 7, it will be observed may be decomposed into $8-1$, putting then $x=8$, $a=1$ in formula (2), we obtain a series the terms of which decrease rapidly.

235. When the terms of the series decrease continually, in general, the more terms we take the nearer we approach the value of the quantity reduced to the series. But if the terms of the series are alternately $+$ and $-$, on stopping at a particular term, we may determine with precision the degree of approximation obtained.

Indeed, let it be supposed, that we have the decreasing series a, b, c, d, e . . . the terms of which are alternately $+$ and $-$, and let x represent the value of this series. If we take any two consecutive sums $a-b+c-d$, $a-b+c-d+e$, for example, with respect to the first the terms, which follow $-d$, will be $\overline{e-f}+\overline{g-h}+\ .\ .\ .$; but since the series is decreasing, the partial differences $\overline{e-f}$, $\overline{g-h}$ will be positive, that is, to obtain the value of x, we must add a positive quantity to $a-b+c-d$; we have therefore

$$a-b+c-d<x$$

With respect to the second sum, the terms which follow $+e$ are $-\overline{h+k}$, $-\overline{l+m}$ . . . ; but the partial differences $-h+k$, $-l+m$ are negative ; to obtain the value of x, we must add therefore a negative quantity to $a-b+c-d+e$, that is, we must subtract a positive quantity from it ; we have therefore $\qquad a-b+c-d+e>x$

The value of x is therefore comprised between the two sums

$$a-b+c-d,\ a-b+c-d+e$$

and since the difference between these is evidently e, it follows, *that the error committed in taking a certain number of terms*

for the value of the quantity reduced to the series is numerically less than the term immediately following the one, at which we stop.

EXAMPLES.

1. To find the approximate fifth root of 39.
Ans. 2.0807 to within .0001.
2. To find the approximate third root of 65.
Ans. 4.02073 to within .00001.
3. To find the approximate fourth root of 260.
Ans. 4.01553 to within .00001.
4. To find the approximate seventh root of 108.
Ans. 1.95204 to within .00001.

236. The binomial formula may also be employed to develop algebraic expressions in series.

Let it be proposed, for example, to develop the expression $\frac{a^2}{a^2-x^2}$ in series.

This expression may be put under the form $a^2(a^2-x^2)^{-1}$, or which is the same thing $a^2 a^{-2}\left(1-\frac{x^2}{a^2}\right)^{-1}$, which becomes by reduction $\left(1-\frac{x^2}{a^2}\right)^{-1}$; developing this last, we have

$$\frac{a^2}{a^2-x^2}=\left(1-\frac{x^2}{a^2}\right)^{-1}=1+\frac{x^2}{a^2}+\frac{x^4}{a^4}+\frac{x^6}{a^6}+\ldots.$$

EXAMPLES.

1. Reduce $\frac{2}{(1-x)^3}$ to a series.
2. Reduce $\frac{1}{(1+x)^5}$ to a series.
3. Reduce $\frac{1}{(a^2-x)^{\frac{1}{3}}}$ to a series.

INDETERMINATE COEFFICIENTS.

237. In the expression $A+Bx+Cx^2+Dx^3+Ex^4+\ldots$, the coefficients $A, B, C, D \ldots$ being independent of x, let

it be required to determine what must be the values of these coefficients, in order that whatever may be the value assigned to x, we may have the equation

$$0=A+Bx+Cx^2+Dx^3 \ldots$$

As the coefficients A, B, C, D . . . in this expression are to be determined, they are on this account called *indeterminate coefficients.*

Since by hypothesis the proposed equation must be verified whatever the value of x, it must be verified, when we assign to x the particular value $x=0$. Putting therefore $x=0$ in the proposed, it is reduced to $0=A$; we have therefore $A=0$; substituting this value of A in the proposed, and dividing both sides by x, we obtain

$$0=B+Cx+Dx^2+\ldots$$

but since this equation must also be verified whatever the value of x, putting $x=0$ we obtain $B=0$. By the same course of reasoning it may be shown also that $C=0$, $D=0$.

In general therefore *if an equation of the form*

$$0=A+Bx+Cx^2+Dx^3+\ldots, \ A, B, C, D \ldots$$

being coefficients independent of x, *may be verified whatever value is given to* x, *each separate coefficient must necessarily be equal to* 0.

This is the principle, upon which the method of indeterminate coefficients is founded. We shall now give some examples of the application of this method.

1. Let there be a dividend $x^2-p'x+p''$, let the divisor be $x-a$, and the quotient $x-q'$; to determine the conditions necessary in order that the division may be exact. Since when there is no remainder, the divisor multiplied by the quotient should produce anew the dividend, we have

$$(x-a)(x-q')=x^2-p'x+p''$$

or performing the multiplication indicated transposing and reducing

$$0=(a+q'-p')x+(p''-aq')$$

whence $\qquad a + q' - p' = 0, \quad p'' - a q' = 0$

wherefore eliminating q'

$$a^2 = a p' - p''$$

In order therefore that the division may be exactly performed, we must have $a^2 = a p' - p''$ or which is the same thing $a^2 - a p' + p'' = 0$.

2. Let it be proposed, as a second example, to determine for A and B values such that we may have the equation,

$$\frac{3 + 5x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

Freeing from denominators transposing and reducing, we have $\qquad 0 = (A + B - 5)x - (2A + 3B + 3)$

whence $\qquad A + B = 5,\ 2A + 3B = -3$

from which we obtain $\qquad A = 18,\ B = -13.$

238. The method of indeterminate coefficients is of great utility in the development of algebraic expressions in series.

Let it be proposed, for example, to develop the expression $\frac{z}{x+z}$ in series, according to the ascending powers of the letter x. In order to this, we assume

$$\frac{z}{x+z} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots.$$

the coefficients A, B, C . . . being independent of x.

Freeing the first member of the proposed from its denominator, transposing, and arranging with reference to x, we obtain

$$0 = \left.\begin{matrix} Az \\ -z \end{matrix}\right| x^0 + \left.\begin{matrix} Bz \\ A \end{matrix}\right| x + \left.\begin{matrix} Cz \\ B \end{matrix}\right| x^2 + \left.\begin{matrix} Dz \\ C \end{matrix}\right| x^3 + \dots.$$

we have therefore the series of equations

$$Az - z = 0,\ Bz + A = 0,\ Cz + B = 0,\ Dz + C = 0\ \dots$$

from the first of which we obtain $A = 1$; substituting this value of A in the second, we have $B = -\frac{1}{z}$, substituting the value of B in the third, we have $C = \frac{1}{z^2}$, and so on.

Substituting the values of A, B, C . . thus obtained in the expression assumed, we have

$$\frac{z}{x+z}=1-\frac{x}{z}+\frac{x^2}{z^2}-\frac{x^3}{z^3}+\frac{x^4}{z^4}-\ldots.$$

Let it be proposed, as a second example, to develop the expression $\frac{1}{3x-x^2}$ in series according to the ascending powers of the letter x.

Putting $\frac{1}{3x-x^2}=A+Bx+Cx^2+Dx^3+\ldots$

we have, freeing from denominators transposing and arranging,

$$0=-1+3Ax+\left.\begin{matrix}3B\\-A\end{matrix}\right|x^2+\left.\begin{matrix}3C\\-B\end{matrix}\right|x^3+\left.\begin{matrix}3D\\-C\end{matrix}\right|x^4+\ldots$$

from which we infer

$$-1=0,\ 3A=0,\ 3B-A=0\ldots$$

but the first of these equations, $-1=0$, is evidently absurd; the proposed therefore does not admit of being developed according to the form required.

The expression $\frac{1}{3x-x^2}$ may, however, be put under the form $\frac{1}{x}\times\frac{1}{3-x}$; let us put therefore

$$\frac{1}{x}\times\frac{1}{3-x}=\frac{1}{x}(A+Bx+Cx^2+Dx^3+\ldots.)$$

from this we obtain

$$0=\left.\begin{matrix}3A\\-1\end{matrix}\right|x^0+\left.\begin{matrix}3B\\-A\end{matrix}\right|x+\left.\begin{matrix}3C\\-B\end{matrix}\right|x^2+\left.\begin{matrix}3D\\-C\end{matrix}\right|x^3+$$

which gives $3A-1=0,\ 3B-A=0,\ 3C-B=0\ldots$

from which we obtain successively

$$A=\frac{1}{3},\ B=\frac{1}{9},\ C=\frac{1}{27},\ D=\frac{1}{81}\ldots$$

we have therefore

$$\frac{1}{3x-x^2}=\frac{1}{x}\left(\frac{1}{3}+\frac{1}{9}x+\frac{1}{27}x^2+\frac{1}{81}x^3+\ldots.\right)$$

or $\dfrac{1}{3x-x^2}=\dfrac{1}{3}x^{-1}+\dfrac{1}{9}x^0+\dfrac{1}{27}x+\dfrac{1}{81}x^2+\ldots$

the proposed contains therefore in its development a term affected with a negative exponent.

EXAMPLES.

1. Let there be the dividend $x^3-p'x^2+p''x-p'''$, the divisor $x-a$; and let the quotient be $x^2-q'x+q''$. To determine the conditions necessary in order that the division may be exact. Ans. $a^3-a^2p'+ap''-p'''=0$.

2. To find for A and B values such that, we may have

$$\frac{7+9x}{(x-5)(x-3)}=\frac{A}{x-5}+\frac{B}{x-3}$$

3. To develop in series the expression $\dfrac{1+2x}{1-x-x^2}$.

4. To develop in series the expression $\dfrac{1}{1+z}$, according to the ascending powers of z, to commence with z^2.

5. To develop in series the expression $\dfrac{a^2}{a^2+2ax-x^2}$.

SECTION IX. PRAXIS.

I. EQUATIONS OF THE FIRST DEGREE.

1. Given $(x+16)^{\frac{1}{2}}=2+x^{\frac{1}{2}}$, to find the value of x.

Ans. $x=9$.

2. Given $x^{\frac{1}{2}}+(x-9)^{\frac{1}{2}}=\dfrac{36}{(x-9)^{\frac{1}{2}}}$, to find the value of x.

Ans. $x=25$.

3. To find the values of x and y in the equations

$$\frac{5x+13}{2}-\frac{8y-3x-5}{6}=9+\frac{7x-3y+1}{3}$$

$$\frac{x+7}{3}:\frac{3y-8}{4}+4x::4:21$$

Ans. $x=5,\ y=4$.

4. To find the values of x, y and z in the equations

$$18x - 7y - 5z = 11$$
$$4\tfrac{2}{5}y - \tfrac{2}{3}x + z = 108$$
$$3\tfrac{1}{2}z + 2y + \tfrac{3}{4}x = 80$$

Ans. $x = 12,\ y = 25,\ z = 6.$

5. To find the values of x, y, z and u in the equations

$$x - 9y + 3z - 10u = 21$$
$$2x + 7y - z - u = 683$$
$$3x + y + 5z + 2u = 195$$
$$4x - 6y - 2z + 9u = 516$$

Ans. $x = 100,\ y = 60,\ z = -13,\ u = -50$

II. EQUATIONS OF THE SECOND DEGREE.

1. Given $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$, to find the values of x.

Completing the square $x^{\frac{6}{5}} + x^{\frac{3}{5}} + \frac{1}{4} = \frac{3025}{4}$

extracting the root $x^{\frac{3}{5}} + \frac{1}{2} = \pm\frac{55}{2}$

from which we obtain $x = 243$ or $(-28)^{\frac{5}{3}}$.

2. Given $x + 5 - (x+5)^{\frac{1}{2}} = 6$, to find the values of x.

Completing the square $x + 5 - (x+5)^{\frac{1}{2}} + \frac{1}{4} = \frac{25}{4}$

extracting the root $(x+5)^{\frac{1}{2}} - \frac{1}{2} = \pm\frac{5}{2}$

from which we obtain $x = 4$, or -1.

3. Given $x^2 - 2x + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 11$, to find the value of x.

Adding 5 to each member

$$x^2 - 2x + 5 + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16$$

completing the square

$$x^2 - 2x + 5 + 6(x^2 - 2x + 5)^{\frac{1}{2}} + 9 = 25$$

extracting the root and reducing, we obtain

$$x = 1 \text{ or } \pm 2\sqrt{15}$$

4. Given $9x+(16x^2+36x^3)=15x^2-4$, to find the values of x.

By transposition &c. $9x+4+2x(9x+4)^{\frac{1}{2}}=15x^2$ completing the square and reducing, we obtain

$$x=\frac{4}{3} \text{ or } -\frac{1}{3}$$

5. Given $\left.\begin{array}{l}x^2+xy=12\\ y^2+xy=24\end{array}\right\}$ to find the values of x and y.

Adding the equations together, we have

$$x^2+2xy+y^2=36, \text{ whence } x+y=\pm 6$$

but $x^2+xy=x(x+y)=\pm 6x$,

whence $\pm 6x=12$; $x=\pm 2$ and $y=\pm 4$

6. Given $\left.\begin{array}{l}x^{\frac{2}{3}}+y^{\frac{2}{3}}=13\\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=5\end{array}\right\}$ to find the values of x and y.

squaring the second equation

$$x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=25$$

but $\quad x^{\frac{2}{3}} \qquad\qquad y^{\frac{2}{3}}=13$

by subtraction $\quad 2x^{\frac{1}{3}}y^{\frac{1}{3}}=12$

subtracting this from the first equation

$$x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=1$$

we have therefore $x^{\frac{1}{3}}+y^{\frac{1}{3}}=5$, $x^{\frac{1}{3}}-y^{\frac{1}{3}}=\pm 1$; from which we deduce $x=27$ or 8, $y=8$ or 27.

7. Given $\left.\begin{array}{l}x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y=19\\ x^2+xy+y^2=133\end{array}\right\}$ to find the values of x and y.

dividing the second equation by the first, we have

$$x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y=7$$

comparing this with the first, we obtain $x+y=13$, $xy=36$; we have therefore $x=9$, $y=4$.

8. Given $x + 4 - 2\left(\frac{x+4}{x-4}\right)^{\frac{1}{2}} = \frac{3}{x-4}$, to find the values of x. Ans. $x = \pm 5$, or $\pm \sqrt{17}$.

9. Given $3x^6 + 42x^3 = 3321$, to find the values of x.

Ans. $x = 3$ or $(-41)^{\frac{1}{3}}$

10. Given $4x^{\frac{1}{3}} + x^{\frac{1}{6}} = 39$, to find the values of x.

Ans. $x = 729$ or $\left(\frac{13}{4}\right)^6$

11. Given $x^2 + 11 = 42 - (x^2 + 11)^{\frac{1}{2}}$, to find the values of x. Ans. $x = \pm 5$ or $\pm (38)^{\frac{1}{2}}$

12. Given $(x - 5)^3 = 40 + 3(x - 5)^{\frac{3}{2}}$, to find the values of x. Ans. $x = 9$ or $-(-5)^{\frac{2}{3}} + 5$.

13. Given $x + (x+6)^{\frac{1}{2}} = 2 + 3(x + 6)^{\frac{1}{2}}$, to find the values of x. Ans. $x = 10$ or -2.

14. Given $9x - 4x^2 + (4x^2 - 9x + 11)^{\frac{1}{2}} = 5$, to find the values of x. Ans. $x = 2$ or $\frac{1}{4}$.

15. Given $\left(x + \frac{8}{x}\right)^2 + x = 42 - \frac{8}{x}$, to find the values of x.

Ans. $x = 4$ or 2.

16. Given $[(x-2)^2 - x]^2 - (x-2)^2 = 90 - x$, to find the values of x. Ans. $x = 6$ or $-1\frac{1}{3}$.

17. Given $x^4\left(1 + \frac{1}{3x}\right)^2 - (3x^2 + x) = 70$, to find the values of x. Ans. $x = 3$ or $3\frac{1}{3}$

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18. Given $\left.\begin{array}{l} x^2 - xy = 54 \\ \text{and } xy - y^2 = 18 \end{array}\right\}$ to find the values of x and y.

Ans. $x = \pm 9$, $y = \pm 3$.

19. Given $\left.\begin{array}{l} 1 + y : x^2 - y^2 :: 1 : 5 \\ \text{and } xy = 21 \end{array}\right\}$ to find the values of x and y.

Ans. $x = 7$ or -3, $y = 3$ or -7.

20. Given $4xy = 96 - x^2y^2$ and $x + y = 6$ } to find the values of x and y.

Ans. $x = 4$ or 2, $y = 2$ or 4.

21. Given $x + y = 6 - (x + y)^{\frac{1}{2}}$ and $x^2 + y^2 = 10$ } to find the values of x and y.

Ans. $x = 3$ or 1, $y = 1$ or 3.

22. Given $x^2 + y^2 - (x + y) = 78$ and $xy + x + y = 39$ } to find the values of x and y.

Ans. $x = 9$ or 3, $y = 3$ or 9.

23. Given $x^2 + y^2 = 34$ and $xy = 15$ } to find the values of x and y

Ans. $x = \pm 5$, $y = \pm 3$.

24. Given $x^3 + y^3 = 189$ and $x^2y + xy^2 = 180$ } to find the values of x and y

Ans. $x = 5$ or 4, $y = 4$ or 5.

25. Given $x^2y + y^2 = 116$ and $xy^{\frac{1}{2}} + y = 14$ } to find the values of x and y.

Ans. $x = 5$, $y = 4$.

26. Given $(x + y)^2 - 3y = 28 + 3x$ and $2xy + 3x = 35$ } to find the values of x and y.

Ans $x = 5$, $y = 2$.

27. Given $x^2 + 4(x^2 + 3y + 5)^{\frac{1}{2}} = 55 - 3y$ and $6x - 7y = 16$ } to find the values of x and y.

Ans. $x = 5$, $y = 2$.

28. Given $x^2 + 3x + y = 73 - 2xy$ and $y^2 + 3y + x = 44$ } to find the values of x and y.

Ans. $x = 4$ or 16, $y = 5$ or 7.

29. Given $\frac{x^2}{y^2} + \frac{4x}{y} = \frac{85}{9}$ and $x - y = 2$ } to find the values of x and y.

Ans $x = 5$, $y = 3$.

30. Given $\left.\begin{array}{r} x+y+(x+y)^{\frac{1}{2}}=12 \\ \text{and } x^3+y^3=189 \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$ or 4, $y=4$ or 6.

31. Given $\left.\begin{array}{r} (6x^{\frac{1}{2}}+6y^{\frac{1}{2}})^{\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}=9-\frac{1}{2}y^{\frac{1}{2}} \\ \text{and } x-y=12 \end{array}\right\}$ to find the values of x and y.

Ans. $x=16$, $y=4$.

32. Given $\left.\begin{array}{r} x^2-xy : xy-y^2 :: 3 : 7 \\ \text{and } xy^2=147 \end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, $y=7$.

33. Given $\left.\begin{array}{r} x^2y+xy^2=6 \\ \text{and } x^3y^2+x^2y^3=12 \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$ or 1, $y=1$ or 2.

34. Given $\left.\begin{array}{r} x^2+2xy+y^2+2(x+y)=120 \\ \text{and } xy-y^2=8 \end{array}\right\}$ to find the values of x and y.

Ans. $x=6$ or 9, $y=4$ or 1.

III. MISCELLANEOUS QUESTIONS.

1. A and B began to trade with equal sums of money. In the first year A gained \$40 and B lost \$40; but in the second A lost one third of what he then had, and B gained a sum less by \$40 than twice the sum that A had lost; when it appeared that B had twice as much money as A. What money did each begin with? Ans. \$320.

2. A footman, who contracted for £8 a year and a livery suit, was turned away at the end of 7 months, and received only £2.3*s*.4*d*. and his livery. What was its value?

Ans. £6.

3. A person engaged to reap a field of 35 acres, consisting partly of wheat and partly of rye. For every acre of rye he received 5 shillings; and what he received for an acre of wheat, augmented by one shilling, is to what he received for an acre of rye as 7 to 3. For his whole labor he received £13. Required the number of acres of each sort.

Ans. 15 acres of wheat and 20 of rye.

4. Two pieces of cloth, of equal goodness but of different lengths, were bought, the one for £5 and the other for £6. 10s. Now if the lengths of both pieces were increased by 10, the numbers resulting would be in the proportion of 5 to 6. How long was each piece, and how much did they cost a yard?

Ans. The price is 5s. and the the lengths are 20 and 26 yds.

5. Two persons A and B have each an annual income of $400. A spends every year $40 more than B, and at the end of 4 years, the amount of their savings is equal to one year's income of either. What does each spend annually?

Ans. $370, and $330 respectively.

6. A cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less than the third per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

Ans. 22, 7, and 12 gallons.

7. A person put out a certain sum to interest for $6\frac{1}{2}$ years at 5 per cent simple interest, and found that if he had put out the same sum for 12 years and 9 months at 4 per cent. he would have received $185 more. What was the sum put out? Ans. $1000.

8. Two persons A and B were partners. A's money remained in the firm 6 years, and his gain was one fourth of his principal, and B's money, which was £50 less than A's had been in the firm 9 months, when they dissolved partnership, and it appeared that if B had gained £6. 5s. less, his gain and principal would have been to A's gain and principal as 4 to 5. What was the principal of each?

Ans. £200 and £150.

9. The estate of a bankrupt valued at $21000 is to be divided among four creditors proportionably to what is due to them. His debts due to A and B are as 2 to 3; B's claims and C's are in the proportion of 4 to 5; and C's and

D's in the proportion of 6 to 7. What sum must each receive? Ans. A \$3200, B \$4800, C \$6000, D \$7000.

10. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 seamen to every three guns and 10 over. Also the whole number of hands was 5 times the number of soldiers and guns together. But after an engagement, in which the slain were one fourth of the survivors, there wanted 5 to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors. Ans. 90 guns, 55 soldiers, and 670 sailors.

11. A shepherd in time of war was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what now remained and $\frac{1}{2}$ of a sheep. After which he had but 25 sheep left. How many had he at first? Ans. 103.

12. A trader maintained himself for 3 years at the expense of \$50 a year; and in each of those years augmented that part of his stock, which was not so expended by $\frac{1}{4}$ thereof. At the end of the third year his original stock was doubled. What was the stock? Ans. \$740.

13. A man and his wife could drink a barrel of beer in 15 days. After drinking together 6 days, the woman alone drank the remainder in 30 days. In what time would either alone drink a barrel?

Ans. the man would drink it in 21 $\frac{3}{10}$ days, and the woman in 50 days.

14 When wheat was 5 shillings a bushel and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye he would have 6 shillings left. How must he lay out his money and fill his sack? Ans. He must buy 9 bushels of wheat, and 12 bushels of rye.

15. A coach set out from Cambridge for London with a certain number of passengers, 4 more being on the outside than within. Seven outside passengers could travel at 2 shillings less expense than four inside. The fare of the whole amounted to £9. But at the end of half the journey, it took up three more outside and one more inside passengers, in consequence of which the fare of the whole became increased in the proportion of 17 to 15. Required the number of passengers, and the fare of the inside and outside.

Ans. There were 5 inside, and 9 outside passengers, and the fares were 18 and 10 shillings respectively.

16. What two numbers are those, whose difference multiplied by the greater produces 40, and by the less 15?

Ans. ± 8 and ± 3.

17. Find two numbers such, that the square of the greater multiplied by the less may be equal to 448, and the square of the less multiplied by the greater may be 392.

Ans. 8 and 7.

18. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time. Ans. 64, 48, 36 and 27 gallons respectively.

19. Find two numbers, whose product is 320, and the difference of their cubes is to the cube of their difference as 61 to unity. Ans. 20 and 16.

20. A farmer has two cubical stacks of hay. The side of one is 3 yards longer than the side of the other; and the difference of their contents is 117 solid yards. Required the side of each. Ans. 5 and 2 yds. respectively.

21. A and B sold 130 ells of silk, of which 40 ells were A's and 90 B's, for 42 crowns. Now A sold for a crown

one third of an ell more than B did. How many ells did each sell for a crown?

Ans. B sold 3 ells, and A $3\frac{1}{3}$ for a crown.

22. A mercer bought a number of pieces of two different kinds of silk for £92. 3*s*. There were as many pieces bought of each kind, and as many shillings paid per yard for them, as a piece of that kind contained yards. Now two pieces, one of each kind, together measured 19 yards. How many yards were there in each?

Ans. 11 yards in one, and 8 in the other.

23. A man playing at hazard won at the first throw as much money as he had in his pocket; at the second throw he won 5 shillings more than the square root of what he then had; at the third throw, he won the square of all he then had; and then he had £112. 16*s*. What had he at first?

Ans. 18 shillings.

24. There are three numbers the difference of whose differences is 3; their sum is 21; and the sum of the squares of the greatest and least is 137. Required the numbers.

Ans. 4, 6 and 11.

25. A grocer sold 80 pounds of mace, and 100 pounds of cloves for £65; but he sold 60 pounds more of cloves for £20, than he did of mace for £10. What was the price of a pound of each?

Ans. the mace 10*s*. and the cloves 5*s*. a pound.

26. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only [illegible] revolutions more than the hind wheel in the same space. Required the circumference of each?

Ans. 4 and 5.

27. A certain sum was to be raised on three estates belonging to A, B, and C, at the rate of one shilling per acre. Now the number of acres A and B had were as 3 to 7; and if the number of acres in the whole were divided by one third of the product of the numbers in the first and third, the

quotient would be $\frac{3}{4}$. Also the sum paid by A and C was 36 shillings less than the sum of three times the money paid by C, and two sevenths of the money paid by B. Of how many acres did each estate consist; and what was the whole sum to be raised?

Ans. A had 12, B 28, C 20 acres; and the sum was £3.

28. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per solid yard, as there were yards in the side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each? Ans. £25 and 16.

29. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

Ans. 5, 10, 20, 40.

30. The sum of three numbers in geometrical progression is 13, and the product of the mean and the sum of the extremes is 30. Required the numbers. Ans. 1, 3, 9.

31. There are three numbers in arithmetical progression, and the square of the first added to the product of the other two is 16; the square of the second added to the product of the other two is 14. What are the numbers?

Ans. 1, 3, 5 or $-5, -3, -1$.

32. There are five whole numbers, the three first of which are in geometrical progression, and the three last in arithmetical progression, the second number being the second difference. The sum of the four last is 40, and the product of the second and last is 64. Required the numbers.

Ans. 2, 4, 8, 12, 16.

33. The number of deaths in a besieged garrison amounted to 6 daily; and allowing for this diminution their stock of provision was sufficient to last for 8 days. But on the evening of the sixth day 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the sixth

day to be sufficient to support 6 men for 6½ days; it is required to find how long it would support the garrison; and the number of men alive when the provisions were exhausted.

Ans. 6 days, and 26 men remained alive when the provisions were exhausted.

34. To find in entire and positive numbers the sides of a rectangle, the surface of which contains four times as many square feet, as its perimeter contains feet.

35. To find three numbers, when the product of each by the difference of the other two is known.

36. Given the sum of two numbers $2s$ and the difference of their cubes $2q$; to find the numbers.

37. The difference of two numbers is $2s$, and the difference of the quotients when the numbers are divided alternately one by the other is $\frac{p}{q}$; to find the numbers.

38. The sum of the surfaces of two rectangles is q, the sum of their bases is a, and if the heights of the rectangles are alternated, the surface of the first will be p, and that of the second q'. To determine the rectangles.

Ans. The base of the first $x = \frac{a[2p + q \pm \sqrt{q^2 - 4pp'}}{2(p + p' + q)}$

39. A merchant deposits in a bank a sum a of dollars a year for n years. How much will he have in the bank at the end of this time, supposing the bank to divide at the rate r per annum.

Ans. $A = \frac{a(r+1)[(r+1)^n - 1}{r}$

2 3

[illegible] was made for skulls [illegible]
[illegible] of brain than [illegible]
[illegible]
[illegible]
[illegible] x, y, z, would harm that head
[illegible]
Equations would [illegible]
[illegible] would be [illegible]

ERRATA.

Page	1,	line	5	for	3 to 5	read	5 to 3
—	1,	—	7	—	3 to 11	—	11 to 3
—	2,	—	3	—	a^2-b^2	—	b^2-c^2
—	16,	—	30	—	$3b^2$	—	$3b^3$.
—	17,	—	26	—	$l+h+k$	—	$l+k+h$
—	97,	—	29	—	$b+h$	—	$b+k$
—	233,	—	13 after whence *insert* log .75 =				
—	234,	—	11 for left read right.				

Note. In the general formulas relative to interest, pages 28, 241 &c. r is considered a decimal.